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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXVI.

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Solved Questions.

1451. (Dr. Hirst, F.R.S.)—Prove that, if

$$F(x, y, a, b, c, f, g, h) = 0$$

be the equation of the n^{th} pedal of the conic

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0,$$

the equation of the $(-n+1)^{th}$ pedal will be

$$F\left(-\frac{x}{r^2}, -\frac{y}{r^2}, \frac{d\Delta}{da}, \frac{d\Delta}{db}, \frac{d\Delta}{dc}, \tfrac{1}{2}\frac{d\Delta}{df}, \tfrac{1}{2}\frac{d\Delta}{dg}, \tfrac{1}{2}\frac{d\Delta}{dh}\right) = 0,$$

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$$+ \sin(\tfrac{1}{2} \cdot \tfrac{1}{3} m \sin^3 n) + \tfrac{1}{2} \cdot \tfrac{1}{3} \sin^3(\tfrac{1}{2} \cdot \tfrac{1}{3} m \sin^3 n) + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} \sin^5(\tfrac{1}{2} \cdot \tfrac{1}{3} m \sin^3 n) + \ldots$$
$$+ \sin(\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} m \sin^5 n) + \tfrac{1}{2} \cdot \tfrac{1}{3} \sin^3(\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} m \sin^5 n)$$
$$+ \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} \sin^5(\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{1}{5} m \sin^5 n) + \ldots$$
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9810. (W. J. C. Sharp, M.A.)—A, B, C are fixed points, P another point; construct the resultant of forces acting along PA, PB, PC, when they are proportional to

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(3) $\frac{1}{PA^2}$, $\frac{1}{PB^2}$, and $\frac{1}{PC^2}$. 85

9926. (Capitaine de Rocquigny.)—Je dispose neuf nombres entiers consécutifs $n-4, n-3, \ldots, n+4$, de manière que le nombre moyen n occupe le centre d'un échiquier de neuf cases, et que les autres nombres se succèdent dans l'ordre de grandeur croissante suivant la loi du cavalier: je forme ainsi le *déterminant du cavalier*. Comment faut-il choisir n pour que la valeur du déterminant soit un carré parfait? 52

11154. (Professor Sylvester.)—Let us understand by the discriminant of a *form* the function whose evanescence is the condition of the equality of the roots in which the coefficient of the term in which only the first and last coefficients of the form appear as unity, and by the discriminant of a rational integral function what the discriminant of its form becomes when the coefficients of that form are the coefficients of the function.

Prove that the discriminant of the function

$$x^n + x^{n-1} + x^{n-2} + \ldots + x + 1,$$

where n is any number greater than unity, is equal to $\frac{(n+1)^{n-1}}{n}$, and show that it is always less than, but continually converges to, a ratio of equality with e/n as n increases indefinitely. 73

11179. (Professor Lancelot.)—On donne, dans un plan, trois droites issues d'un même point et faisant entre elles un angle de 120°; lieu géométrique des points de l'espace tels que le triangle obtenu en joignant les projections de ces points sur les trois côtés ou sur leurs prolongements, ait une aire donnée. 34

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11477. (R. Knowles, B.A.)—A circle is drawn through PQ, a chord of a rectangular hyperbola centre O, and its pole, cutting the curve again in CD; the circle OPQ cuts the curve again in C'D'; T, T' are the poles of CD, C'D' with respect to the hyperbola. Prove that (1) T, T', O are collinear; (2) TO = T'O. 28

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11528. (Editor.)—Required a general investigation of the following problem, proposed by the Editor many years ago as Quest. 1072, and solved, when reproposed as Quest. 4390, on p. 87 of Vol. XLII.:—"Two gamesters, A and B, play together, A having the power to fix the stakes. Whenever A loses a game, he increases his last stake by a shilling for the next game, and diminishes it by a shilling after each gain. When they leave off playing, A has gained £13; and, had each won the same number of games, A would still, by following this method in regulating his stakes, have won 10 shillings. The first stake having been 30 shillings, find the number of games each won." [This question is reproposed for sake of its relation to "the Wells Martingale," or "steady and certain system" of winning at Monte Carlo, which is thus set forth in the second leader of the *Daily Telegraph* for April 18th, 1892: "You always play on the colour which turned up last, and you always add one unit after every loss, and risk one less after every time you have won." Under Quest. 4596, a similar problem is solved, on p. 35 of Vol. XXIII. An investigation is desired that shows the relation of the *general* problem to "the Wells Martingale."] 117

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11652. (Professor Morley.)—Denoting the absolute value of $x+iy$, *i.e.*, $+(x^2+y^2)^{\frac{1}{2}}$, by $|x+iy|$, prove

$$|z+(z^2-c^2)^{\frac{1}{2}}| + |z-(z^2-c^2)^{\frac{1}{2}}| = |z+c| + |z-c|. \quad \ldots \quad 99$$

11707 & 11720. (A. E. Jolliffe, M.A.)—From two points B, C, tangents are drawn to a conic S, and these four tangents, together with the polars of B and C with respect to S, all touch a conic α. Similarly the pairs of points CA, AB determine the conics β and γ, respectively. Prove, by pure geometry, that, if A lie on α, then B lies on β, and C on γ.

(11720.) In Quest. 11707, prove also that the sides of the triangle ABC meet the conic S in three pairs of points, such that the three pairs of straight lines joining them to the opposite vertices intersect by threes in two points. 43

12542. (Professor Scheffer.)—According to Bessel the ratio of the square of the polar diameter of the earth to that of the equatorial diameter is ·9933254. Find (1) at what altitude the angle made by a body falling to the earth with a perpendicular to the surface is greatest; and (2) also this maximum angle. 68

12547. (Professor Newton.)—The axes of two right cylinders whose bases are circles of 4 and 6 inches radius respectively intersect at right angles; compute to four decimal places the lengths of the curves of intersection of the two surfaces. 70

12598. (W. J. Dobbs, M.A.)—It is desired to place a given triangle in such a position that the times of descent from rest down its three sides, considered as smooth material straight lines, may be equal. Show that this cannot be done unless the triangle is right-angled; and that, if this condition be satisfied, the hypotenuse of the triangle may be placed in any position, and the plane of the triangle must then be turned about its hypotenuse until it is at right angles to the vertical plane which contains the hypotenuse. 50

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12657. (R. Knowles, B.A.)—From a point T on the line $x = 3a$ tangents are drawn to meet the parabola $y^2 = 4ax$ in PQ; the circle TPQ cuts the curve again in CD; CD meets the normals at P, Q in R, R', respectively; prove that the diameters of the parabola through RR' bisect the chords of curvature at P, Q, respectively. 29

12681. (Professor Wright.)—Find when the Dog Star and the Sun rise together in latitude 42° 30′ North = λ: given the R.A. of Sirius = 6 h. 40 m. 30 sec., and its dec. = 16° 32′ 56″ S. 57

12696. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{2.3.4.5.6.7} + \frac{1}{8 \ldots 13} + \ldots = \frac{\pi}{80\sqrt{3}} + \frac{1}{120} - \frac{2\log 2}{45},$$

$$\frac{1}{1.3.5.7.9.11} + \frac{1}{13 \ldots 23} + \ldots = \frac{23\pi}{23040\,(17 + 9\sqrt{3})},$$

$$\frac{1}{1.2.3} + \frac{1}{5.6.7} + \frac{1}{9.10.11} + \frac{1}{13.14.15} + \ldots = \tfrac{1}{4}\log 2,$$

$$\frac{1}{1.2.3} \cdot \frac{1}{5.6.7} + \frac{1}{9.10.11} \cdot \frac{1}{13.14.15} + \ldots = \frac{\pi}{480\,(3\sqrt{2} + 4)}. \quad \ldots \quad 37$$

12712. (Professor Symmond.)—Show that, of all curves of a given length, traced on one plane between two given points, and made to revolve around a common axis situated in that plane, the catenary generates a minimum area. 85

12760. (John Griffiths, M.A.)—Show that in a right-angled triangle the nine-point circle is a G.B. circle (see *Proceedings of the London Mathematical Society*, June, 1894). 67

12780. (Professor Hudson, M.A.)—A unit is divided into three parts so that by taking two of them I may get either $\frac{3}{7}$ or $\frac{9}{10}$ of the unit; what fraction shall I get if I take the third pair. Express each of the parts as a percentage of the whole. 37

12785. (Professor Davison.)—For the purpose of locating the most eligible point for a county seat, it is required to determine the centre of a county whose dimensions are as follows:—Beginning at the south-west corner, thence east 15 miles, thence north $33\frac{3}{8}$ miles, then west 6 miles to north end of a meridian line running south through the county; thence south-westerly to a point being 6 miles west from the meridian line, and $9\frac{3}{8}$ miles south of its north end, thence south 3 miles, thence west 3 miles, thence south 21 miles to place of beginning..... 35

12855. (Professor Zerr.) — One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometres per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 100 grams mass, and moving with a velocity each of 10 kilometres per second; collisions take place, and finally, after a number of encounters, the average kinetic energy of each of the two thousand balls is the same. Show that this is $2{\cdot}75\ (10)^{11}$ in the centimetre-gram-system. 83

12868. (H. J. Woodall, A.R.C.S.)—Elucidate the following extract from the "Bemerkungen," which Gauss' Editor (Schering) has collected at the end of Vol. II. of the *Werke* (p. 523). "Es bezeichnen darin

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[2], [5], [13], ... (18), (57), (239), ... die Bögen der Cotangenten 1, 2, $\frac{3}{2}$, ... 18, 57, 239, ... Mit Hülfe der Tafeln ist durch Zurlegung von $18+i$, $57+i$, $239+i$ in ihre complexe Primfactoren

$$(18) = 2[2]-2[5]-[13], \quad (57) = -[2]+3[5]-[13],$$
$$(239) = 3[2]-4[13] \text{ gefunden."}$$

The tables referred to are explained thus:—"Die Tafeln *zur Cyklotechnie* geben für 2452 Zahlen von der form a^2+1, a^2+4, a^2+9, ... a^2+81 die sämmtlichen ungeraden Primtheiler p neben den zugehörigen a und zwar in solchen Fällen, wo die Primtheiler alle unter 200 liegen, nur dann werden a^2+1 u.s.f. zerlegen genannt."

Extract from table a^2+1.

18	5 . 5 . 13
57	5 . 5 . 5 . 13
239	13 . 13 . 13 . 13 120

12960. (W. C. Stanham.)—OA is a horizontal line on an inclined plane; OB is drawn perpendicular to OA, along the plane. At O there is a fine smooth hole through which passes a string connecting two particles P (mass m) and Q (mass $m \sin\alpha$), α being the inclination of the plane. P is held at A and then let go. The first time its path meets OB it cuts it at an angle β, and at the point B. Show that, if the curved line AB revolve round OB, the volume generated

$$= \tfrac{2}{3}\pi OA \,.\, OB^2/1+(1+\sin\alpha)\cot^2\beta. \ldots\ldots\ldots\ldots 55$$

12963. (R. Chartres.)—In an ellipse, if $ae = b/e$, and p the extremity of the latus rectum, prove that (1) the tangent at P intercepted by the axes will be the minimum tangent to the ellipse and the minimum straight line through P; (2) it will be divided in extreme and mean ratio at P; (3) the angle PTC $= \tan^{-1}e = \cos^{-1}e =$ eccentric angle of P; (4) the normal PG will go through B, the extremity of the minor axis; (5) the radius of curvature at P = BG = semi-conjugate diameter to CP; (6) BP will be cut in extreme and mean ratio at G; and (7), if BCT be the surface of a rectangular lake, and a person at B could row b miles per hour and walk a miles, then to get to T in the shortest time he must row along the normal chord to G and walk GT, requiring e^{-1} and e hours, respectively, and arriving at T in e^{-3} hours. 26

12966. (F. A. L. Kitchin, B.A.)—In the method of "clearing" a lunar distance used in Inman's *Nautical Tables*, we have

$$2\cos A = (\sin z \sin z')/(\cos a \cos a'),$$

where z, z' are the true zenith distances of a star and the moon's centre, and a, a' the apparent altitudes of the same. Deduce a simple formula for A, and find A's limiting values. 26

12973. (Professor Radhakrishnan, B.A.)—Cut off the nth part of a triangle by a straight line drawn from a given external point. 29

12980. (Professor Shields.)—A railroad turn-table 100 feet long is balanced upon a pivot in the centre of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk? 36

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12988. (J. J. Barniville, B.A.)—Sum the series

$$\frac{1}{1.2}+\frac{1}{6.7}+\frac{1}{11.12}+\ldots. \ldots\ldots\ldots\ldots 54$$

12994. (I. Arnold.)—PT, *pt* are tangents to a parabola at P, *p*, the extremities of the ordinates MP, *mp* on opposite sides of the axis, and the points P, *p* are joined; prove that

(1) $\tan \mathrm{TPM} \times pm = \tan tpm \times \mathrm{PM}$; (2) $\tan \mathrm{TPM} - \tan tpm = 2 \tan p\mathrm{PM}$; and show their utility in practical gunnery, MP being the greater ordinate. 28

12995. (G. W. L. West.)—Give solutions of $3n^2-3n+1 = \square$.... 53

13004. (Professor Chakrivarti.)—A tower, whose height is 60 feet, is situated on the side of a hill which rises due south at an angle of 45°. What length of ladder will be necessary to scale the tower, supposing the foot of the ladder to be placed at a distance of 30 feet in a south-west direction from the base of the tower? 45

13010. (Professor Shields.)—Four tortoises, A, B, C, D, start from the same place, on a level plain. B and C travelled at the same rate; the travelling rates of A and B were as 17 to 18 respectively, and the travelling rates of C and D were as 18 to $19\frac{1}{17}$ respectively; or the rates of all were as 17 to 18 to $19\frac{1}{17}$ respectively. A could travel one mile in 1 hr. 26 min. 24 sec., and A travelled due west a certain distance, thence due north a certain distance. B travelled due north the cube of A's distance west *plus* his distance west; thence B travelled due east the fifth power of A's distance north, *plus* his distance north; C travelled due east a certain distance; thence C travelled due south a certain distance, and D travelled due south the cube of C's distance east *plus* his distance east; thence D travelled due west the fifth power of C's distance south, *plus* his distance south, when it was found that the sum of the distances B and D travelled north and south was 467280 inches, and the sum of the distances B and D travelled east and west was 1949266638 inches, and that the product of the distance A and C travelled east and west *plus* the square of the difference of these distances *plus* 1, was 4720, and that the product of the distances A and C travelled north and south *plus* the square of the difference, squared, *plus* the product multiplied by the square of the difference, *plus* 1, was 19689562. Find how far each travelled in all, and how long it required for B and D each to make the entire trip from starting place to the end. 86

13018. (D. Biddle.) — Given the sum of two unknown quantities and the sum of their reciprocals, each sum being real and positive, find, numerically, the cube-root of each original quantity, or, failing that, the sum of their cube-roots, when the product of the two given sums is less than 4. 55

13026. (Rev. T. Roach, M.A.)—On the Bedford Level three poles of equal height are set up in a straight line at distances of 660 yards, and a man looks from the top of the first pole at the top of the third. Show that, owing to the rotundity of the earth, the top of the second pole will be above his line of sight, and calculate the difference approximately, taking the radius of the earth as 3960 miles. 62

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13028. (Rev. T. Wood Robinson, R.N.) — Two ships A and B are at sea; if A be faster than B, then, as long as A keeps B on a certain fixed bearing, (1) B can never close on A, (2) if B tries to close by keeping A dead ahead, show their distance apart will remain constant, and (3) find their paths in this case. 81

13030. (R. F. Davis, M.A.)—Two circles intersect orthogonally at B, C, and EF is the diameter of the first circle passing through the centre of the second. Through any point U on the second circle draw EU, FU meeting the first again in P, Q. Prove that a third circle can be inscribed in the quadrilateral PBQC, whose centre is at U. Hence show that the rectangle contained by the diagonals of any quadrilateral inscribed in one fixed circle, and circumscribed to another fixed circle, is constant. (Compare Quest. 12974.) [The Proposer states that there appeared in the *Reprint*, many years ago, a semi-geometrical proof (by himself) of the relation connecting the distance between the centres and the radii of two circles such that quadrilaterals could be inscribed in the one and circumscribed to the other. It was reprinted in Casey's *Trigonometry*, and this is an *extension* of the idea of the proof.] 73

13038. (Professor Gopalachanar.) — Prove that the difference in length of the two loops of the limaçon $r = 2\cos\theta + 1$ is 8. 71

13041. (Professor Wolstenholme.) — If $u = 0$ be the rational equation of the second degree (coordinates rectangular), and if the equation

$$\frac{d^2u^{\frac{1}{2}}}{dx^2} + \frac{d^2u^{\frac{1}{2}}}{dy^2} + \frac{d^2u^{\frac{1}{2}}}{dz^2} = 0$$

represent a paraboloid (a true paraboloid excluding cylinders), prove that it will be a paraboloid of revolution. [It would be interesting to see this proved directly from the general equation, where

$$u \quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2a''x + 2b''y + 2c''z + d\,;$$

viz., if the equation

$$(ax + hy + gz + a'')^2 + (hx + by + fz + b'')^2 + (gx + fy + cz + c'')^2 = (a + b + c)\,u$$

represent a paraboloid, it must be one of revolution.] 39

13042. (Professor Sanjána.)—In an isosceles triangle ABC, right-angled at C, CA is produced to D, so that CD = AB; DB is joined, and the angle DBE is made = BDC. Prove that CE is one-third of DE. [This is the geometrical proof of $\tan^{-1}\sqrt{2} - \cot^{-1}\sqrt{2} = \sin^{-1}\frac{1}{3}$.] ... 49

13046. (Professor Jules.) — Une colonne verticale de hauteur h est surmontée d'un mât vertical de hauteur l. On demande à quelle distance horizontale du pied de la colonne il faudra se placer pour voir la colonne et le mât sous des angles égaux. 37

13047. (Professor Scheffer.)—The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault. 40

13068. (Rev. T. Wiggins, S.J.)—A person in a boat w miles from the nearest point of the beach wishes to reach in the shortest time a place x miles from that point along the shore. Supposing he can walk y miles an hour, but pull only at the rate of z miles an hour, where must

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he land? Explain the absence of x in the answer, and the obvious discrepancy when x is less than the distance between the nearest point and the landing point. 65

13074. (Professor Genese, M.A.)—Two circles orthogonal to a fixed circle meet in real or imaginary points, according as the join of their centres meets the fixed circle in imaginary or real points, respectively; also, the cosine of the angle, real or imaginary, at which the two circles cut $= \frac{1}{2}(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})$, where μ is the cross-ratio in which the join of centres is divided by the fixed circle. 33

13075. (Professor Zerr.)—I have a circular-section basin 12 inches in perpendicular height; the diameters are as follows:—at base, 2 inches; 1-inch perpendicular height, 6 inches; 2-inches perpendicular height, 18 inches; 3-inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After rain, the water in the basin is 6 inches deep; what was the rainfall? 84

13077. (Professor Sanjána.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \log(1+\cos x)\,\frac{dx}{\cos x} = \frac{\pi^2}{8} = \frac{3}{2^2} - \frac{5}{6^2} + \frac{11}{30^2} - \frac{13}{42^2} + \frac{19}{90^2} - \ldots$$

$$\int_{-\infty}^{0} \cos x \log(x^2)\,dx = \pi; \quad \text{and find} \quad \int_0^{\infty} \frac{dx}{\left[(1+x^2)(1+mx^2)\right]^{\frac{1}{2}}}, \quad \text{where } m > 0 \text{ and } < 1. \ldots\ldots 34$$

13082. (Professor Ignacio Beyens.)—On donne les côtés d'un quadrilatère inscriptible. Calculer la longueur de la troisième diagonale, c'est-à-dire de la droite qui joint les intersections des côtés opposés. 30

13100 & 13151. (M. Brierley.)—Given the difference of the segments of the base made by the perpendicular from the vertex, the difference of the angles at the base and the rectangle under the sum of the sides and the line bisecting the vertical angle, to construct the triangle.... 41

13116. (R. F. Davis, M.A.)—If A, B, C, D be four points on a parabola, the portions of the diameter bisecting AB, intercepted respectively by AC, BD and AD, BC, will be equal. 48

13123. (W. Stanham.)—A centre of force C, which attracts matter with a force inversely as the square of the distance, moves in a hyperbola. A thin brittle straight rod has its extremities fixed at the foci. Show that, if the rod breaks (owing to the attraction) at D, CD is parallel to an asymptote of the hyperbola. 56

13126. (Rev. G. Richardson, M.A.)—If a circle be drawn touching two sides of a triangle so that the chord of contact passes through the incentre or one of the excentres, prove that the circle touches the circumcircle. 46, 94

13128. (M. Brierley.)—Find the sum, to infinity, of

$$\left\{\frac{1}{3} + \frac{1}{2.3}\right\} - \left\{\frac{1}{5} + \frac{1}{2.5} + \frac{1}{3.5} + \frac{1}{4.5}\right\} + \left\{\frac{1}{7} + \frac{1}{2.7} + \frac{1}{3.7} + \frac{1}{4.7} + \frac{1}{5.7} + \frac{1}{6.7}\right\} - \&c.$$

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13131. (Professor Neuberg.)—On donne un triangle ABC. Par un point quelconque M de BC on mène MP perpendiculaire à AB, MQ perpendiculaire à AC, et l'on trace PQ. Démontrer que la perpendiculaire abaissée de M sur PQ enveloppe une parabole. 49

13133. (Professor Sanjána.)—Let the sides AB, CB of a triangle be produced to B_1, B_2, respectively, and let lines parallel to the symmedian of B be placed in the angles ABB_2, CBB_1; call the locus of their mid-points the ex-symmedian of B; proceed similarly for the other angles. Then (1) prove that the ex-symmedians of two angles and the symmedian of the third meet in a point; call the three points thus obtained K_1, K_2, K_3, the symmedian point being K; (2) show that the circle described with K_1 as centre and K_1B or K_1C as radius cuts two sides produced at the extremities of a diameter; so for K_2 and K_3; (3) if BK_1, CK_1 meet this circle again in β_1, γ_1, and AB, AC meet $\beta_1\gamma_1$ produced in P_1, Q_1, find the circum-radius R_1 of AP_1Q_1, also of BP_2Q_2, CP_3Q_3; (4) prove that $R_1^{-1}+R_2^{-1}+R_3^{-1} = R^{-1}$; and (5) prove that the distance

$$KK_1 = 2a^2bcm_1/(b^2+c^2-a^2+2bc),$$

where m_1 is the median corresponding to A; also prove similar expressions for KK_2, KK_3. 41

13134. (Professor Zerr.)—A man raises 1 chicken the first year, 6 the second, 35 the third, 180 the fourth, 921 the fifth, 4,626 the sixth, 23,215 the seventh, 116,160 the eighth, and so on. How many does he raise the twentieth year, and how many in the 20 years? 82

13136. (Professor Barisien.)—Le cercle de courbure C en un point quelconque M d'une ellipse rencontre la normale en M en un second point M'. Le diamètre du cercle C perpendiculaire à MN' rencontre ce cercle en N et N'. Montrer que chacune des courbes lieux des points M', N, N' a une aire équivalente à celle de l'ellipse. 47

13139. (Professor Watson.)—Find the average area of all right-angled triangles having a given hypotenuse. 60

13141. (Professor Myers.)—A flexible cord of given length is suspended from two points whose coordinates are (x, y) and (x', y'); find what the condition of the chord must be in order that it may hang in the arc of a circle. 47

13143. (Professor Matz.)—A., B., and C. can walk at the rate of 3, 4, and 5 miles an hour. They start from Washington at $m = 1$, $n = 2$, $p = 3$ o'clock p.m., respectively. When B. overtakes A., he is ordered by A. back to C. When will B. and C. meet? Suppose B. had ordered A. back to C., when would A. and C. meet? In case all three continue walking ahead, at what time will they meet? 53

13144. (Editor.)—If ABC be a triangle in a circle, AD the bisector of the angle A, AP a tangent meeting BC at P, PQR a perpendicular to AD meeting the circle in Q, R, prove that

$$AQ^2 = BQ \cdot CQ, \quad AR^2 = BR \cdot CR. \ldots\ldots 48$$

13147. (R. F. Davis, M.A.)—ABCD is a quadrilateral inscribed in a circle centre O. If, upon AB, CD, points P, S be taken, respectively, such that OP = OS, and PS produced meet the circle in X, Y, and AC,

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BC, AD, BD in Q, U, V, K, respectively, prove that UX = VY. PX = SY, and PQ = RS. 50

13148. (W. C. Stanham.)—P and Q are the points of contact of the tangents from (x', y') to the ellipse $x^2/a^2 + y^2/b^2 = 1$. Prove that, if R be a point on the ellipse such that its eccentric angle ϕ satisfies the relation $x'^2 \sec^2\phi - y'^2 \operatorname{cosec}^2\phi = a^2 - b^2$, with the further condition that

$$ay' \operatorname{cosec}\phi - bx' \sec\phi \text{ and } ax' \sec\phi - by' \operatorname{cosec}\phi + a^2 - b^2$$

are of the same sign, the difference of the arcs PR and QR (measured in opposite directions round the ellipse) is either zero or equal to the length of a known straight line. 45

13149. (R. Knowles, B.A.)—CP, CD are equi-conjugate radii of an ellipse; tangents at P and D intersect in T; prove that the radii of the in-circle and circum-circle of the triangle TPD are respectively

$$2^{-\frac{1}{2}}ab\left[(a^2+b^2)^{\frac{1}{2}} + a\right]^{-1} \text{ and } (a^2+b^2) \div 2^{\frac{3}{2}}b;$$

a and b being the semi-axes of the ellipse. 39

13150. (Rev. G. Richardson, M.A.)—The locus of the orthocentre of the triangle formed by three normals to a parabola, two of which are fixed, is another parabola. 51

13153. (Rev. T. W. Robinson, R.N.) — POP′ and QOQ′ are two straight lines inclined to one another at an angle C; OP and OQ are equal to $\tan\frac{1}{2}b$ and $\tan\frac{1}{2}a$, and OP′ and OQ′ to $\cot\frac{1}{2}b$ and $\cot\frac{1}{2}a$, where a, b, and C are two sides and included angle of a spherical triangle ABC. Prove the angles PQ′P′ and Q′P′Q are equal to the angles B and A of the spherical triangle, and the angle PQ′O is equal to half the spherical excess. 109

13154. (I. Arnold.) — Divide a given circle into two segments, so that the sum of the squares inscribed in them may be equal to one half the square on the diameter. 32

13162. (Professor Neuberg.)—Construire un triangle ABC, connaissant les longueurs AB, AC de deux côtés et sachant que les symédianes issues de B et C sont rectangulaires. 75

13163. (Professor Sanjána.)—On the same circle as base and on the same side of it are described a hemisphere and a cone of vertical angle $(2\alpha < 90°)$; find the volume of the hemisphere outside the cone, and of the cone outside the hemisphere. 77

13168. (Professor Nath Coondoo.)—Prove that the whole arc of the curve $8a^2y^2 = x^2(a^2 - 2x^2)$ is πa; and for the part included in the positive quadrant the centre of gravity of the area is $\left(\frac{3\pi a}{16\sqrt{2}}, \frac{1}{20}a\right)$; the centre of gravity of the arc is $\left(\frac{5a}{3\sqrt{2\pi}}, \frac{a}{4\pi}\right)$; the centre of gravity of the volume generated by revolution about the axis of x is $\left(\frac{5a}{8\sqrt{2}}, 0\right)$; and that of the area of the surface generated is $\left(\frac{3a}{5\sqrt{2}}, 0\right)$. 72

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13170. (Editor.)—If $y = x^2 - 1$, prove that

$$y\frac{d^{n+1}y^n}{dx^{n+1}} = n(n+1)\frac{d^{n-1}y^n}{dx^{n-1}};$$

$$y\frac{d^{m+2}y^n}{dx^{m+2}} + 2(m-n+1)x\frac{d^{m+1}y^n}{dx^{m+1}} = (m+1)(2n-m)\frac{d^m y^n}{dx^m}. \quad \ldots \quad 74$$

13171. (D. Biddle.)—Prove that

$$\int_0^a x^n (a-x)^r \, dx = \frac{n!\, r!\, a^{n+r+1}}{(n+r+1)!},$$

a being any quantity, and n, r integers of any magnitude. 64

13172. (R. F. Davis, M.A.)—Given S a fixed focus, and OX, OY, fixed directrices; find the envelope of PQ where P, Q are variable points taken one on each directrix equidistant from S. [The Proposer desires a verification of his result. Draw SM, SN perpendicular to the directrices; then the envelope is the first negative pedal with respect to S of the rectangular hyperbola passing through ONSM. The particular case in which S lies on the bisector of YOX should be noticed.] 65

13174. (W. C. Stanham.)—Show that $\int_0^\infty y\,dx$ has the same value for the two curves $xy^2 = f(x+2a)$ and $xy^2 = f(x+a^2/x)$. 67

13175. (R. Knowles, B.A.)—PQ is the chord of contact of tangents to a parabola, focus F, drawn from T; show that (1) the centre O of the circle TPQ lies on the circle PQF; (2) OF and TF are at right angles; (3) if PQ passes through a fixed point on the axis, the envelope to the tangent to the circle at O is a parabola. 76

13177. (M. Brierley.)—From a given point draw two tangents to a given circle, and join the points of contact. From the centre of the circle draw any radius (produced, if necessary), perpendicular to which draw a line from the given point, and produce it indefinitely. If, then, lines be drawn from the points of contact of the tangents to intersect in the extremity of the radius before drawn, they will intercept on the indefinite line a portion of it of a constant magnitude. 59

13178. (Rev. T. Wiggins, S.J.)—A candidate is examined in four papers, to each of which is assigned m marks as a maximum. His total on the four papers is $\frac{1}{3}\frac{1}{3}m$. In how many ways can this occur? Supposing there were n papers with a maximum of m marks each, and that his total was q: can a solution be found to this? 71

13181. (H. W. Segar, B.A.)—Show that

$$\left(a^2\frac{d}{da} + b^2\frac{d}{db} + \ldots\right) H_n = (n+1)H^{n+1} - H_1 H_n,$$

where H_n is the sum of the homogeneous products of the quantities $a, b, c, \ldots$ of degree n. Also find the result of n operations. 77

13183. (A. Gordon, M.A.)—Two parallel planes are drawn in an ellipsoid, each parallel to one of the principal sections. Two lines are described with these sections as bases and the centre of the ellipsoid as common vertex. Required the relation between x, the distance apart of

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the planes, and V, the volume of the ellipsoid contained between the cones ($x \propto V$). 84

13184. (P. W. Flood.)—In a circle 100 yards in diameter, three equal circles are inscribed touching each other and the outer circle. Find the diameter of a circle inscribed in the space at the centre. ... 80

13185. (C. Bickerdike.)—C is a fixed and D a variable point on a circle whose centre is E. Produce CD to F, so that DF equals the diameter of the circle. Find the radius of curvature of the locus of F. 78

13186. (H. Orfeur.) — If the denominator of the penultimate convergent of a continued fraction be equal to the numerator of the ultimate convergent, the partial quotients equidistant from the ends are equal. 61

13188. (Rev. T. Roach, M.A.)—Prove geometrically that

$$\cos^{-1}\tfrac{8}{17} + \sin^{-1}\tfrac{5}{13} = \cos^{-1}\tfrac{21}{221}. \text{ ..... } 88$$

13191. (Professor Neuberg.) — Soient O le centre du cercle circonscrit au triangle ABC, et I, I_a, I_b, I_c les centres du cercle inscrit et des cercles exinscrits. On prend sur les côtés BA, CA, vers le sommet A, les longueurs $BB' = CC' = BC$, et aussi, dans la direction opposée, $BB'' = CC'' = BC$. Cela posé: (1) Les droites B'C', B''C'', B'C'', B''C' sont respectivement perpendiculaires aux droites OI, OI_a, OI_b, OI_c; (2) les rayons des cercles circonscrits aux triangles AB'C', AB''C'', AB''C' sont respectivement égaux à OI, OI_a, OI_b, OI_c. 79

13193. (Professor Cochez.)—Limite de

$$\left(1 + \frac{2}{\sqrt{(m^2+1)}}\right)^{\sqrt[3]{(3m^3+1)}} \quad \text{pour } m = \infty. \text{ ..... } 70$$

13195. (Professor Honey.)—Let *ab* and *cd* be respectively the major and minor axes of an ellipse, and let *a* be the angle which a diameter *th* forms with the major axis; find the length of this diameter. ... 59

13199. (Professor Laisant.)—Désignons par (OP, OQ) l'ellipse construite sur les demi-diamètres conjugués OP, OQ. Soient A', B', C' trois foyers des ellipses (OB, OC), (OC, OA), (OA, OB) respectivement. Démontrer que les trois ellipses (OA, OA'), (OB, OB'), (OC, OC') sont homofocales. 61

13202. (Professor Izitzéica.) — On considère un point M sur une parabole P. La normale en M, à P, rencontre l'axe de la parabole en N. La perpendiculaire en N, à l'axe, rencontre P en Q et R. Trouver le lieu du centre de gravité du triangle formé par les points de contact des normales à P en M, Q, R avec la développée de la parabole. 58

13203. (Editor.) — If, in a triangle ABC, right-angled at A, the bisector of the angle A meet the side *a* (*i.e.*, BC) in Q and the circumcircle in P, prove that $PA : PQ = (b+c)^2 : a^2$. 86

13204. (Rev. T. Wiggins, S.J.)—O is the centre of a circle, Q is a point without it, P, P' are the points of contact of the tangents from Q, R, R' are points where the circle is cut by any secant through Q; show that the tangents at R, R' meet on P, P' produced. 57

13207. (R. F. Davis, M.A.)—Prove that (1) the sides of a triangle

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are the roots of the equation $x^3-2sx^2+(s^2+r^2+4Rr)x-4Rrs=0$; and (2) the cosines of the angles are the roots of

$$4R^2x^3-4R(R+r)x^2+x(s^2+r^2-4R^2)-s^2+(2R+r)^2=0,$$

so that the triangle is acute-angled or obtuse-angled according as s is greater or less than $2R+r$. 98

13208. (M. Brierley.)—Given a line drawn from the vertex to the middle of the base, and one of the angles at the base, to construct the triangle when the area is a maximum. 64

13211. (R. Knowles, B.A.)—Rectangular hyperbolas are drawn to have contact of the second order with the parabola $y^2=4ax$, at a point P $(am^2, 2am)$; prove that the locus of the mid-points of their common chords is the parabola $y^2-2ax-2amy+2a^2m^2=0$. 103

13212. (I. Arnold.)—ABCD is a quadrilateral, and AC, BD the diagonals intersecting in E. Make BF = DE, and AR = CE; and let Et_1 be $\frac{1}{3}$ of EF, and $Et=\frac{1}{3}$ of ER. Through t and t_1 draw ts and t_1s parallel to the diagonals; and prove that s is the centre of gravity of the quadrilateral. 95

13214. (R. F. Muirhead.)—Prove the following formulæ :—

$$\frac{(2n)!}{(2r)!\,(2n-2r)!}=n!\left\{\frac{1}{r!\,(n-r)!}+\frac{2^2}{2!\,(r-1)!\,(n-r-1)!}+\frac{2^4}{4!\,(r-2)!\,(n-r-2)!}+\dots\right\},$$

$$\frac{(2n)!}{(2r+1)!\,(2n-2r-1)!}=n!\left\{\frac{2}{1!\,r!\,(n-r-1)!}+\frac{2^3}{3!\,(r-1)!\,(n-r-2)!}+\frac{2^5}{5!\,(r-2)!\,(n-r-3)!}+\dots\right\},$$

where the series in each case is to be continued to the term which contains 0! (read as unity) as a factor in its denominator. 69

13215. (H. W. Curjel, M.A.) — Prove that, n, p, q being any positive quantities,

$$\frac{1}{\Gamma(p+q)}-n\frac{q}{\Gamma(p+q+1)}+\frac{n(n-1)}{1.2}\frac{q(q+1)}{\Gamma(p+q+2)}-\frac{n(n-1)(n-2)}{1.2.3}\frac{q(q+1)(q+2)}{\Gamma(p+q+3)}+\dots=\frac{\Gamma(p+n)}{\Gamma(p)\,\Gamma(p+q+n)}.$$

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13216. (Franz Rogel.)—Prove that

$$\phi(2)\left|\frac{n}{2}\right|+\phi(3)\left|\frac{n}{3}\right|+\phi(4)\left|\frac{n}{4}\right|\dots+\phi(n-1)\left|\frac{n}{n-1}\right|+\phi(n)\left|\frac{n}{n}\right|=\binom{n}{2},$$

wherein $\phi(m)$ denotes the quantity of numbers $<m$ prime to m, and $\left|\frac{n}{r}\right|$ the greatest integer $\geqq\frac{n}{r}$. 62

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13220. (Professor Neuberg.) — On projette le point M en A′ et B sur les côtés de l'angle AOB, en C′ et D′ sur les côtés de l'angle CO′D, en E′ et F′ sur les côtés de l'angle EO″F. Trouver (1) le lieu d'un point M; tel que les droites A′B′, C′D′, E′F′ concourent en un même point P; (2) lieu de ce point P. 94

13222. (Professor Coondoo.)—If on a straight line of length $a+b+c$ there be measured at random two lengths a, b, the chance of their having a common part not greater than d is $\frac{(c+d)^2-c^2}{(c+a)(c+b)}$, $(d<a$ or $b)$; the chance of their not having a common part greater than d is $\frac{(c+d)^2}{(c+a)(c+b)}$, and the chance of the smaller b lying altogether within the larger a is $\frac{a-b}{a+c}$. 114

13224. (Professor Krishmachandra De.) — If ρ_1, ρ_2, ... ρ_n be the distances of a point P in the plane of a regular polygon of n sides from the vertices, prove that

$$\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\ldots+\frac{1}{\rho_n^2}=\frac{n}{r^2-a^2}\cdot\frac{r^{2n}-a^{2n}}{r^{2n}-2r^na^n\cos n\theta+a^{2n}},$$ 118

13228. (Professor Barisien.)—L'équation du 4e degré

$$(b^2z^2-a^2)^2-4ab\,(bz^2-a)(az-b)=0$$

est décomposable en un produit de deux facteurs du second degré. Des racines de cette équation, deux sont toujours réelles; les deux autres, toujours imaginaires. 102

13231. (Professor Finkel.)—A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to α, and on walking to B, a distance a, directly down the hill towards the foot of the pole, the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole? 91

13233. (Editor.)—If θ, ϕ be the angles subtended by two conjugate diameters at a point in the curve of an ellipse, prove that $\cot^2\theta+\cot^2\phi$ is constant. 97

13234. (Hugh MacColl, B.A.) — Prove the following formulæ in symbolic logic:—(1) $\alpha^x\beta^x=(\alpha\beta)^x$; (2) $\alpha^x\alpha^y=\alpha^{x+y}$; (3) $(\alpha^x)^y=\alpha^{xy}$;

(4) $\frac{x}{\alpha}\,.\,\frac{y}{\alpha}=\frac{x+y}{\alpha}$; (5) $\frac{\alpha}{x}\,.\,\frac{\alpha}{y}:\frac{\alpha}{xy}$; (6) $\frac{\alpha}{xy}=\alpha_{xy}\left(\frac{\alpha}{x}+\frac{\alpha}{y}\right)$.

Defs.—The symbol α^x is short for $\alpha+x'$, and asserts that either α is true or x false—possibly both. The symbol $\alpha\beta$ (or $\alpha\,.\,\beta$) asserts the truth of both α and β. The symbol $\alpha:\beta$ (or α_β) asserts that whenever α is true β is true. The *fractional* symbol α/β is short for $\alpha_\beta\,(\alpha'_\beta)'$; it asserts what α_β asserts, and *also* that β is *not* always true when α is *not* true. 95

13238. (Rev. J. Cullen, B.A.)—Solve the difference equation

$$u_{x+1}u_x+x^2\,(u_{x+1}+u_x)+x^4+4x^2-1=0.$$ 104

13240. (R. Chartres.)—If the sides of a triangle be in arithmetical

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progression, and the bisector of the greatest angle $(1/m)^{th}$ of the perimeter, find in lowest integers (1) the sides and bisector, and (2) the sides, bisector, and m; and (3) the greatest angle in each case. 102

13243. (R. Knowles, B.A.)—From a point T in the directrix of a parabola, whose focus is F, and vertex A, tangents are drawn to meet the curve in P, Q; show that (1) $TP \cdot TQ = (FT)^3/AF = \sqrt{(AF \cdot PQ^3)}$; (2) if a third tangent at t meets these in p, q, $(pq)^2 = PQ \cdot Ft$. 91

13245. (H. J. Woodall, A.R.C.S.)—Prove that $2^{155}-1$ is divisible by $(31)^2$. 97

13247. (R. F. Davis, M.A.)—Prove that the cotangent of the Brocard-angle (1) of any triangle is $\frac{1}{2}\{s/r-(4R+r)/s\}$, (2) of the triangle formed by joining the points of contact of the sides of the original triangle with the inscribed circle is $(4R+r)/s$. 98

13250. (Professor Neuberg.)—Les carrés des côtés d'un triangle ABC sont racines de l'équations $A_0x^3+3A_1x^2+3A_2x+A_3=0$. Trouver l'angle de Brocard de ABC. 103

13254. (Professor Ramaswami Aiyar.)—ABC is a given acute-angled triangle. Points P, Q, R are taken at random in the sides BC, CA, AB; and perpendiculars are drawn at P, Q, R to those sides. Show that the probability that the triangle formed by these perpendiculars lies entirely within the triangle ABC is $\frac{1}{2}(\sin A \sin B \sin C)/(\cot A+\cot B+\cot C)$. 89

13258. (Professor Siverly.)—Of all cones which develop an ellipsoid and have their bases in the tangent plane at a given point P, and are of the same altitude, that is the least which has its vertex in the diameter through P; and of all which have their vertices in this diameter, that is the least whose axis is twice this diameter. 113

13261. (Professor Lilley.)—A drawbridge, a feet in length, moves uniformly about a centre axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its centre, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a = 320$ and $n = 2$. 115

13264. (W. E. K. Hare, B.A., B.C.S.)—(1) Prove that all chords of an ellipse which subtend a right angle at a fixed point O on the ellipse pass through a fixed point Z on the normal at O. (2) Determine the locus of Z for different positions of O on the ellipse. 92

13265. (C. E. Hillyer.)—Having given a straight line in a circle, construct a second straight line equal to the first, so that the lines through the middle point of the first parallel to the second and through the middle point of the second parallel to the first may meet on the circumference. 115

13274. (Rev. T. Wiggins, B.A.)—Inscribe the maximum ellipsoid in a cone of given volume..... 113

13278. (Professor Neuberg.)—Trouver le lieu d'un point tel que les

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aires de ses triangles podaires par rapport à deux triangles donnés ABC, A'B'C' aient une somme donnée. 108

13279. (Professor Sanjána, M.A.)—Given the two segments of an altitude made by the orthocentre and the corresponding base of a triangle; construct it. 105

13281. (Professor Bernés.)—Dans un triangle ABC, le pied de la hauteur abaissée de A étant A', et P, Q désignant les projections de B, C sur le rayon AO du cercle ABC, le cercle A'PQ a pour centre C milieu de BC. 113

13288. (H. MacColl, B.A.)—Prove the following formulæ in symbolic logic:—

(1) $(\alpha+\beta)_x : a_x+\beta_x : (a\beta)_x$, (2) $x_{a\beta} : x_a+x_\beta : x_{a+\beta}$,

(3) $(\alpha+\beta)_{ax} = a_{ax}+\beta_{ax}$, (4) $x_a x_{a\beta} : a_{a\beta}$,

(5) $x/a \,.\, x/o\beta : a/o\beta$, (6) $a/\beta = a_\beta \epsilon_{a\beta}$, (7) $a_\beta = (\epsilon : \beta^a)$.

Defs.—A : B : C is short for (A : B)(B : C); a_β is the denial of a_β; $a/o\beta$ is the denial of a/β: the symbol ϵ denotes a statement *always true*, so that ϵ_a asserts that a is *always true*, and ϵ_{a_a} asserts that a is *not* always true. [See Quest. 13234.] 119

13292. (F. H. Jackson.)—Prove that

$$1+n\,.\,x_1y_{-1}+\frac{n\,.\,n-1}{2!}x_2y_{-2}+\ldots = \frac{(y)_{-n}}{(x+y)_{-n}},$$

where
$$x_n = \frac{\Pi(x)}{\Pi(x-n)} \quad \ldots\ldots(1),$$

analogous to the binomial expansion

$$1+n^1xy^{-1}+\frac{n\,.\,n-1}{2!}x^2y^{-2}+\ldots = \frac{(y)^{-n}}{(x+y)^{-n}} \quad \ldots\ldots(2),$$

No. (1) being a variation of Vandermonde's theorem, as (2) is of the binomial theorem; n is not necessarily an integer in (1). 118

13296. (Rev. T. Wiggins, B.A.)—Given a circle of radius R, on any diameter BB' produced take a point A, such that the distance from A to O, the centre, is equal to a. Draw a secant AMM' making with AO an angle θ. Let β, γ be the angles which the radii OM, OM' make with the diameter BB'. It is required (1) to calculate $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ in terms of R, a, θ; (2) to show that the product $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ is constant, whatever be the value of θ; (3) to find the minimum value of the sum $\tan^2\frac{1}{2}\beta+\tan^2\frac{1}{2}\gamma$. 112

13298. (R. F. Davis, M.A.)—Employing the usual notation for a triangle, note the following expressions

$$OI^2 = R^2-2Rr = R(R-2r), \quad 4IN^2 = (R-2r)^2,$$
$$OH^2 = 9R^2+8Rr+2r^2-2s^2 = (R-2r)(9R+26r)-2\Delta^2,$$
$$9IG^2 = 5r^2-16Rr+s^2 = \Delta^2-16r(R-2r),$$
$$IH^2 = 4R^2+4Rr+3r^2-s^2 = 4(R-2r)(R+3r)-\Delta^2,$$
$$\&c., \quad \&c.,$$

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where $\Delta^2 = s^2 - 27r^2$; and show generally that any symmetrical function of the sides or angles of a triangle may be expressed in terms of s, R, r. [In case of equilateral triangle $R = 2r$, and $\Delta^2 = 0$.] 111

13299. (P. W. Flood.)—Between a tangent drawn at one end of the segment of a circle and the chord produced draw another tangent divided by the point of contact into two segments that shall be in a given ratio. 111

13300. (R. Knowles, B.A.) — The circle of curvature centre O is drawn at a point P of a conic, M is the mid-point of their common chord; the diameter of the conic through M meets the normal at P in Q; prove (1) that Q is the point through which pass all chords of the conic that subtend a right angle at P; (2) the ratio OQ : OM is constant. 110

ERRATUM.

Page 76 Question 3984, at foot. The latter part of result should read

$$\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.k^n\right)^2.$$

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

11197. (W. J. GREENSTREET, M.A.)—An equiangular polygon is circumscribed to a parabola: prove that (1) the consecutive radii vectores drawn from the focus to the vertices and the points of contact make equal angles with each other; (2) the polygon is described in an hyperbola, one of the foci of which is the focus of the parabola; (3) the polygon, the vertices of which are the points of contact of the first polygon, can be inscribed in an ellipse, which has as one of its foci the focus of this parabola; and (4) the joins of the vertices of the first polygon to the points of contact of the second with the ellipse pass through the focus.

Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

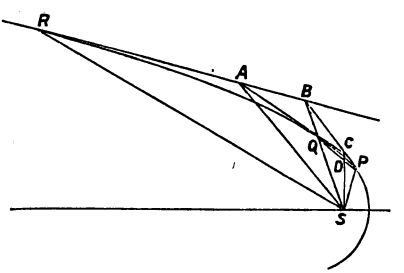
(1) Let PCB, CQA, BAR be three consecutive tangents at P, Q, R, and let S be the focus. Then

$\angle CSB = \angle CAB = \angle ACB = \angle BSA.$

But $\angle QSC = \angle BCA$;

$\therefore$ SQB is a straight line, and

$\angle PSC = \angle CSQ = \angle QSA = \angle ASR = \beta$ (say).



(2) Let the coordinates of P and Q be $r_1,\ \theta-\beta$; $r_2,\ \theta+\beta$; and let $SC = r$.

Then
$$r = (r_1 r_2)^{\frac{1}{2}} = \frac{4a}{\left[\left\{1+\cos(\theta-\beta)\right\}\left\{1+\cos(\theta+\beta)\right\}\right]^{\frac{1}{2}}}$$
$$= \frac{4a\sec\beta}{1+\sec\beta\cos\theta};$$

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therefore the vertices of the circumscribing polygon lie on the hyperbola $r = \frac{4a \sec \beta}{1 + \sec \beta \cos \theta}$, with focus S and eccentricity $\sec \beta$.

(3) If we take r, ϕ as current coordinates, the equation to PQ is

$$r \{\cos \phi (\cos \theta + \cos \beta) + \sin \theta \sin \phi\} = 4a \cos \beta,$$

which is the equation to the tangent to the ellipse

$$r = \frac{4a \cos \beta}{1 + \cos \theta \cos \beta} \quad \text{at point} \quad \left(\frac{4a \cos \beta}{1 + \cos \theta \cos \beta}, \ \theta \right);$$

therefore PQ touches this ellipse at D, where it cuts CS.

12963. (R. Chartres.)—In an ellipse, if $ae = b/e$, and p the extremity of the latus rectum, prove that (1) the tangent at P intercepted by the axes will be the minimum tangent to the ellipse and the minimum straight line through P; (2) it will be divided in extreme and mean ratio at P; (3) the angle PTC = $\tan^{-1} e = \cos^{-1} e$ = eccentric angle of P; (4) the normal PG will go through B, the extremity of the minor axis; (5) the radius of curvature at P = BG = semi-conjugate diameter to CP; (6) BP will be cut in extreme and mean ratio at G; and (7), if BCT be the surface of a rectangular lake, and a person at B could row b miles per hour and walk a miles, then to get to T in the shortest time he must row along the normal chord to G and walk GT, requiring e^{-1} and e hours, respectively, and arriving at T in e^{-3} hours.

Solution by the Proposer, Professor Gopalachanar, *and others.*

Since $ae = b/e$,

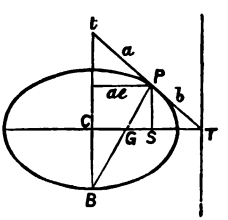
$\therefore \quad 1 - e^2 = e^4, \quad$ or $\quad a^2 - b^2 = ab,$

and $\quad \tan \text{PTC} = e = \cos \text{PTC}$

$= \cos$ of the auxiliary angle.

Also $Pt = \frac{ae}{\cos \text{PTC}} = a$, and $PT = b$.

Therefore Tt is the minimum tangent, &c.; and, similarly, the other properties follow.



12966. (F. A. L. Kitchin, B.A.)—In the method of "clearing" a lunar distance used in Inman's *Nautical Tables*, we have

$$2 \cos A = (\sin z \sin z') / (\cos a \cos a'),$$

where z, z' are the true zenith distances of a star and the moon's centre, and a, a' the apparent altitudes of the same. Deduce a simple formula for A, and find A's limiting values.

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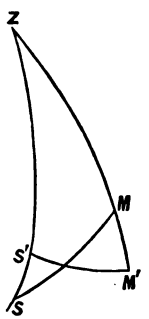
Solution by C. Bickerdike, Professor Krishnamacharry, *and others.*

Let S, M be the true places of the star and moon in the vertical circles SZ, MZ, making the angle MZS; then, because parallax and refraction take place in vertical circles, M will be above M′, and S will be below S′.



Let D, d; a, α; H, h be the true and apparent distances, altitudes, and apparent altitudes.

Then, in $\triangle$SZM,

$$\cos \mathrm{SZM} = \frac{\cos \mathrm{D} - \sin \mathrm{A} \,.\, \sin a}{\cos \mathrm{A} \,.\, \cos a} = \frac{\cos \mathrm{A} - \sin \mathrm{H} \,.\, \sin h}{\cos \mathrm{H} \,.\, \cos h};$$

$$\therefore \quad \cos \mathrm{D} = (\cos d - \sin \mathrm{H} \,.\, \sin h) \frac{\cos \mathrm{A} \,.\, \cos a}{\cos \mathrm{H} \,.\, \cos h} + \sin \mathrm{A} \,.\, \sin a;$$

$$\therefore \quad \sin^2 \tfrac{1}{2}\mathrm{D} = \cos^2 \tfrac{1}{2}(\mathrm{A}+a) - (\cos \tfrac{1}{2}\mathrm{H} + h + d) \cos \tfrac{1}{2}(\mathrm{H}+h-d) \frac{\cos a \,.\, \cos \mathrm{A}}{\cos h \,.\, \cos \mathrm{H}}.$$

Now, by taking both sides of this equation from 1, and making the right-hand side $= \sin^2 \phi$, we get

$$\sin^2 \tfrac{1}{2}\mathrm{D} = \cos^2 (\mathrm{A}+a) \cos^2 \phi; \quad \therefore \quad \sin \tfrac{1}{2}\mathrm{D} = \cos (\mathrm{A}+a) \cos \phi,$$

a simple formula for calculating the true from the apparent distance.

Again, let D = 0; then

$$\cos (\mathrm{A}+a) \cos \phi = 0, \quad \text{or} \quad \cos \tfrac{1}{2}\mathrm{A} \,.\, \cos \tfrac{1}{2}a - \sin \tfrac{1}{2}\mathrm{A} \,.\, \sin \tfrac{1}{2}a = 0;$$

$$\therefore \quad \tan \tfrac{1}{2}\mathrm{A} = \cot \tfrac{1}{2}a,$$

when both bodies are on the same vertical circle.

This process is given by Peacock in his *Astronomy*, p. 446; the same formula is also given by Bell in Chambers's *Mathematics*, Part II., p. 292.

11272. (G. H. Bryan, M.A.)—Give a linkage of jointed bars for dividing any angle into any number (n) of equal parts, the number of bars in the linkage not exceeding $2n$.

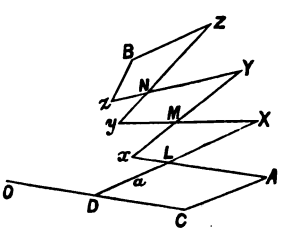
Solution by H. W. Curjel, M.A.; Professor Lampe; *and others.*

The straight lines in the figure represent bars which are jointed together wherever they cross.

$$\mathrm{BZ} = \mathrm{ZN} = \mathrm{NY} = \mathrm{YM} = \mathrm{MX} = \mathrm{XL} = \mathrm{LA} = \mathrm{DC},$$

$$\mathrm{B}z = z\mathrm{N} = \mathrm{N}y = y\mathrm{M} = \mathrm{M}x = x\mathrm{L}, \quad \mathrm{AC} = \mathrm{DL},$$

and $\quad \mathrm{OD} : \mathrm{OC} = x\mathrm{L} : \mathrm{LA}$ $\quad (= 1 : 2 \text{ in figure}).$



It is clear that, if O be placed at an angle, and A moved along one of the

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bounding straight lines of the angle till B falls on the other, Z, N, Y, M, X, L will lie on the six lines which divide $\angle AOB$ into seven equal parts; the linkage is clearly quite general, and will be similar for any odd value of n. If n is even (6 in figure), the parts BZ, Bz, zN may be omitted.

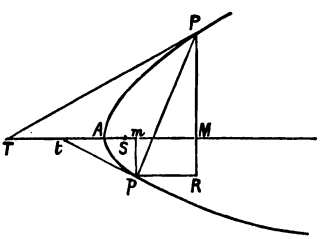
12994. (I. Arnold.)—PT, pt are tangents to a parabola at P, p, the extremities of the ordinates MP, mp on opposite sides of the axis, and the points P, p are joined; prove that

(1) $\tan \mathrm{TPM} \times pm = \tan tpm \times \mathrm{PM}$; (2) $\tan \mathrm{TPM} - \tan tpm = 2 \tan p\mathrm{PM}$;

and show their utility in practical gunnery, MP being the greater ordinate.

Solution by Rev. J. L. Kitchin, M.A.; Professor Lampe; *and others.*

$$\mathrm{Tan\,TPM} = \frac{\mathrm{TM}}{\mathrm{MP}} = \frac{2\mathrm{AM}}{\mathrm{MP}} = \frac{4\mathrm{AM}\,.\,\mathrm{AS}}{2\mathrm{MP}\,.\,\mathrm{AS}} = \frac{\mathrm{MP}^2}{2\mathrm{AS}\,.\,\mathrm{MP}} = \frac{\mathrm{MP}}{2\mathrm{AS}}.$$



So $\tan tpm = \dfrac{\mathrm{MP}}{2\mathrm{AS}}$;

$\therefore$ $\mathrm{MP} \tan \mathrm{TPM} = \mathrm{MP}\,.\,\tan tpm.$

Again, $$\tan \mathrm{TPM} - \tan tpm = \frac{\mathrm{MP} - mp}{2\mathrm{AS}}.$$

But $\mathrm{PM}^2 = 4\mathrm{AS}\,.\,\mathrm{AM}$, $pm^2 = 4\mathrm{AS}\,.\,\mathrm{A}m$, $\mathrm{PM}^2 - pm^2 = 4\mathrm{AS}\,.\,m\mathrm{M}$;

$$\therefore \quad \mathrm{PM} - pm = \frac{4\mathrm{AS}\,.\,\mathrm{M}m}{\mathrm{PM} + pm} = \frac{4\mathrm{AS}\,.\,p\mathrm{R}}{\mathrm{PR}} = 4\mathrm{AS} \tan p\mathrm{PM};$$

$$\therefore \quad \tan \mathrm{TPM} - \tan tpm = 2 \tan p\mathrm{PM}.$$

[Of the gunnery value of this, no correspondent knows anything.]

11477. (R. Knowles, B.A.)—A circle is drawn through PQ, a chord of a rectangular hyperbola centre O, and its pole, cutting the curve again in CD; the circle OPQ cuts the curve again in C′D′; T, T′ are the poles of CD, C′D′ with respect to the hyperbola. Prove that (1) T, T′, O are collinear; (2) TO = T′O.

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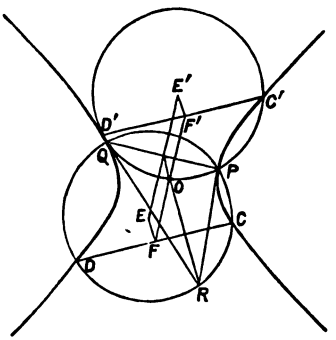
Solution by H. W. CURJEL, M.A.; Prof. KRISHMACHANDRA DE; *and others.*



Let R be the pole of PQ, F and F′ the middle points of CD, C′D′; E, E′ the centres of circles PQR, PQD′C′.

Since angles POQ, PRQ are together equal to two right angles, the circles PQR, PQC′ are equal.

Also, PQ, CD are equally inclined to the axes, and PQ, C′D′ are equally inclined to the axes;

∴ CD, C′D′ are parallel;

∴ TT′O are collinear.

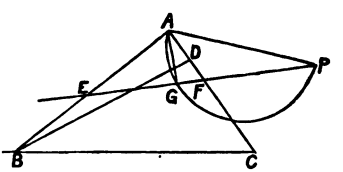
Again, F′O and RO are equally inclined to the axes, and PQ and RO are equally inclined to the asymptotes, which are at right angles, and bisect the angles between the axes; ∴ F′O is perpendicular to PQ; ∴ FF′ is parallel to EE′; but E′F′, EF are parallel;

∴ EF = E′F′; ∴ C′D′ = CD;

∴ CD and C′D′ are equidistant from O; ∴ TO = T′O.

12973. (Professor RADHAKRISHNAN, B.A.)—Cut off the nth part of a triangle by a straight line drawn from a given external point.

Solution by I. ARNOLD, Professor KRISHNAMACHARRY, *and others.*



Let ABC be the triangle, and P the external point.

Divide AC in D, so that ABD may be the nth part of the △ABC, and through P draw PFE cutting AB in E and AC in F, so that the rectangle AE . AF = AB . AD.

It is evident that the △AEF is equal to ABD; ∴ &c.

12657. (R. KNOWLES, B.A.)—From a point T on the line $x = 3a$ tangents are drawn to meet the parabola $y^2 = 4ax$ in PQ; the circle TPQ cuts the curve again in CD; CD meets the normals at P, Q in R, R′, respectively; prove that the diameters of the parabola through RR′ bisect the chords of curvature at P, Q, respectively.

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Solution by Rev. J. L. KITCHIN, M.A.; Professor MOREL; *and others.*

Let $\angle COR = \theta$, OR′ radius of the circle $= r$, $OB = a$, and ρ, θ the polar coordinates of the circle, O pole; then

$$r^2 = (a+r)^2 + \rho^2 - 2\rho(a+r)\cos\theta,$$

whence ρ_1, ρ_2, or OR, OR′ are

$$\rho_1 = (a+r)\cos\theta - \{r^2 - (a+r)^2\sin^2\theta\}^{\frac{1}{2}},$$

$$\rho_2 = (a+r)\cos\theta + \{r^2 - (a+r)^2\sin^2\theta\}^{\frac{1}{2}}$$

$$\therefore \quad RR' = 2\{r^2 - (a+r)^2\sin^2\theta\}^{\frac{1}{2}}.$$

Now $$ABRR' = \sin\theta\,(a+r)^2\{\lambda\cos\theta + (\lambda^2 - \sin^2\theta)^{\frac{1}{2}}\},$$

where $$\lambda = r/(a+r);$$

$$\therefore \quad \sin\theta\{\lambda\cos\theta + (\lambda^2 - \sin^2\theta)^{\frac{1}{2}}\} = \text{a maximum};$$

$$\cos\theta\{\lambda\cos\theta + (\lambda^2 - \sin^2\theta)^{\frac{1}{2}}\} - \lambda\sin^2\theta - \frac{\sin^2\theta\cos\theta}{(\lambda^2 - \sin^2\theta)^{\frac{1}{2}}} = 0;$$

$$\therefore \quad \lambda^2(1 - 4\sin^2\theta\cos^2\theta)(\lambda^2 - \sin^2\theta) = \lambda^4 - 4\lambda^2\sin^2\theta + 4\sin^4\theta;$$

whence, reducing and dividing by $\lambda^2 - 1$, we get

$$\lambda^2 - 4\lambda^2\cos^2\theta + 4\sin^2\theta\cos^2\theta = 0, \text{ which determines } \theta.$$

Now $$\lambda^2 = 4\cos^2\theta(\lambda^2 - \sin^2\theta);$$

$$\therefore \quad \lambda = 2\cos\theta(\lambda^2 - \sin^2\theta)^{\frac{1}{2}};$$

$$\therefore \quad r = 2\cos\theta\{r^2 - (a+r)^2\sin^2\theta\}^{\frac{1}{2}} = 2\cos\theta \,.\, RR'.$$

13082. (Professor IGNACIO BEYENS.) — On donne les côtés d'un quadrilatère inscriptible. Calculer la longueur de la troisième diagonale, c'est-à-dire de la droite qui joint les intersections des côtés opposés.

Solution by W. E. HEAL, M.A.; Professor SANJÂNA; *and others.*

Let

$AB = a$, $BC = b$, $CD = c$, $DA = d$.

Draw CH making

$$\angle FCH = \angle DEH;$$

then, the triangles BFC, DFA being similar,

$$\therefore \quad BC : CF : FB = DA : AF : FD;$$

whence

$$CF = b(bc + ad)/(d^2 - b^2),$$

$$FD = d(ab + cd)/(d^2 - b^2).$$

In like manner,

$$CD : DE : EC = AB : BE : EA;$$

whence $EC = c(bc + ad)/(a^2 - c^2)$, $EB = a(ab + cd)/(a^2 - c^2)$.

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The similar triangles CHF, DEF; CHE, BFE give

$$FH : FC = FD : FE, \quad HE : EC = EB : EF;$$

$$\therefore \quad (FH + HE)\,EF = EF^2 = FC \cdot FD + EC \cdot EB$$

$$= (ab + cd)(bc + ad)\left[bd(a^2 - c^2)^2 + ac(b^2 - d^2)^2\right] / (a^2 - c^2)^2 (b^2 - d^2)^2.$$

9746. (Professor Déprez.)—Dans le triangle ABC, on mène la médiane et la symmédiane issues du sommet A; ces droites rencontrent BC en M, N et la circonférence ABC en M′, N′. Démontrer: (1) que les droites de Simson des points M′, N′ sont respectivement perpendiculaires aux droites AN′, AM′; (2) que leurs distances à M′ et N′ sont égales à S^2/m^2n, S^2/m^3, S désignant l'aire ABC, et m, n les longueurs AM, AN.

Solution by Professors Zerr, Swaminatha Aiyar, *and others.*

Let BC, AB be the axes of x, y; then we have

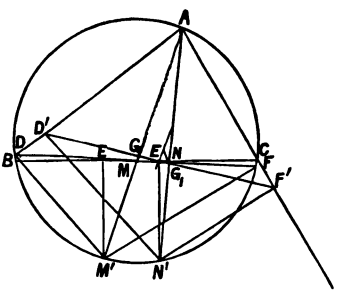
$$x^2 + 2xy\cos B + y^2 - ax - cy = 0$$

as the equation to the circle; the equations to AM, AN are

$$2x/a + y/c = 1,$$

$$\frac{(b^2 + c^2)\,x}{ac^2} + \frac{y}{c} = 1;$$

and the coordinates of M′ are



$$\left(\frac{a^3 - 2a^2c\cos B + 2ac^2}{a^2 - 4ac\cos B + 4c^2}, \quad -\frac{a^2c}{a^2 - 4ac\cos B + 4c^2}\right).$$

The coordinates of E, the foot of the perpendicular from M′ on BC, are

$$\left(\frac{a^3 - 3a^2c\cos B + 2ac^2}{a^2 - 4ac\cos B + 4c^2}, \quad 0\right).$$

The coordinates of D, the foot of the perpendicular from M′ on AB, are

$$\left(0, \quad \frac{a^3\cos B - 2a^2c\cos^2 B + 2ac^2\cos B - a^2c}{a^2 - 4ac\cos B + 4c^2}\right);$$

therefore the equation to DF, the Simson line of the point M′, is

$$y = \frac{a^3\cos B - 2a^2c\cos^2 B + 2ac^2\cos B - a^2c}{a^2 - 4ac\cos B + 4c^2} - \frac{(a^3\cos B - 2a^2c\cos^2 B + 2ac^2\cos B - a^2c)\,x}{a^3 - 3a^2c\cos B + 2ac^2};$$

$$\therefore \quad y = d - mx, \text{ suppose.}$$

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Also, $y = c - \dfrac{(b^2+c^2)\,x}{ac} = d' - m'x$ is the equation to AN;

$\therefore\ 1 + (m+m')\cos B + mm' = 0$; $\therefore$ DF is perpendicular to AN.

The coordinates of N′ are $\left(\dfrac{2ac^2}{b^2+3c^2-2ac\cos B},\ \dfrac{c^3-b^2c-2ac^2\cos B}{b^2+3c^2-2ac\cos B}\right)$.

The coordinates of E′, the foot of the perpendicular from N′ on BC, are

$$\left(\frac{2ac^2\sin^2 B + c^3\cos B - b^2c\cos B}{b^2+3c^2-2ac\cos B},\ 0\right).$$

The coordinates of D′, the foot of the perpendicular from N′ on AB, are

$$\left(0,\ \frac{c^3-b^2c}{b^2+3c^2-2ac\cos B}\right);$$

therefore the equation to D′F′, the Simson line of the point N′, is

$$y = \frac{c^3-b^2c}{b^2+3c^2-2ac\cos B} - \frac{(c^3-b^2c)\,x}{2ac^2\sin^2 B + c^3\cos B - b^2c\cos B} = f - nx, \text{ suppose.}$$

$y = c - 2cx/a = f' - n'x$ is the equation to AM;

$\therefore\ 1 + (n+n')\cos B + nn' = 0$; $\therefore$ D′F′ is perpendicular to AM.

Coordinates of m' may be written $\left(\dfrac{abc}{2mn},\ -\dfrac{a^2c}{4m^2}\right) = (x_1, y_1)$;

$$N' = \left(\frac{ac^2}{2m^2},\ -\frac{a^2c}{4m^2}\right) = (x_2, y_2);\quad n = 2bcm/(b^2+c^2);$$

$$y = \frac{bm\,(a^2+c^2-b^2) - a^2cn}{4m^2n} - \frac{\{2bn\,(a^2+c^2-b^2) - 2a^2cn\}\,x}{an\,(3b^2-a^2+c^2)},$$ equation to DF;

therefore $A_1x + B_1y + C_1 = 0$ is equation to DF.

$$y = \frac{c^3-b^2c}{4m^2} - \frac{2\,(c^3-b^2c)\,x}{a\,(3c^2-a^2+b^2)},$$ equation to D′F′;

$\therefore\ A_2x + B_2y + C_2 = 0$ is equation to D′F′;

$\therefore$ distance from M′ to DF is $-\dfrac{(A_1x_1 + B_1y_1 + C_1)\sin B}{(A_1^2 + B_1^2 - 2A_1B_1\cos B)^{\frac{1}{2}}} = S^2/m^2n$;

distance from N′ to D′F′ is $-\dfrac{(A_2x_2 + B_2y_2 + C_2)\sin B}{(A_2^2 + B_2^2 - 2A_2B_2\cos B)^{\frac{1}{2}}} = S^2/m^3$.

13154. (I. Arnold.)—Divide a given circle into two segments, so that the sum of the squares inscribed in them may be equal to one half the square on the diameter.

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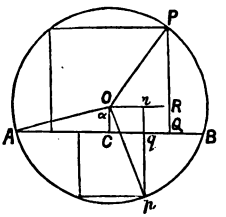
Solution by Professors SANJÁNA, GOPALACHANAR, *and others.*



Let the dividing line AB subtend an angle 2α at the centre, OP $= a$, CQ $= x$; then

$$OP^2 = OR^2 + RP^2$$

gives $\quad x^2 + (2x - a\cos\alpha)^2 = a^2,$

i.e., $\quad 5x^2 - 4xa\cos\alpha - a^2\sin^2\alpha = 0.$

From this, we have

$$x = \tfrac{1}{10}\left\{4a\cos\alpha \pm (16a^2\cos^2\alpha + 20a^2\sin^2\alpha)^{\frac{1}{2}}\right\};$$

rejecting the negative value,

$$PQ = 2x = \tfrac{1}{5}\left\{4a\cos\alpha + 2a(4 + \sin^2\alpha)^{\frac{1}{2}}\right\}.$$

So also $\quad pq = \tfrac{1}{5}\left\{2a(4 + \sin^2\alpha)^{\frac{1}{2}} - 4a\cos\alpha\right\}.$

Hence $\quad PQ^2 + pq^2 = \tfrac{1}{5}\left\{8a^2(4 + \sin^2\alpha) + 32a^2\cos^2\alpha\right\}$

$$= \tfrac{8}{25}a^2(5 + 3\cos^2\alpha).$$

If this $= 2a^2$, we find $12\cos^2\alpha = 5$, *i.e.*, $\alpha = \cos^{-1}\sqrt{5}/\sqrt{12}$, which shows how the construction may be made.

13074. (Professor GENESE, M.A.)—Two circles orthogonal to a fixed circle meet in real or imaginary points, according as the join of their centres meets the fixed circle in imaginary or real points, respectively; also, the cosine of the angle, real or imaginary, at which the two circles cut $= \tfrac{1}{2}(\mu^{\frac{1}{2}} + \mu^{-\frac{1}{2}})$, where μ is the cross-ratio in which the join of centres is divided by the fixed circle.

Solution by W. E. HEAL, Professor NATH COONDOO, *and others.*

Let $x^2 + y^2 - r^2 = 0$ be the fixed circle; $x'y'$, $x''y''$ the centres of the orthogonal circles. Their equations and radical axis are

$$x^2 + y^2 - 2x'x - 2y'y + r^2 = 0, \quad x^2 + y^2 - 2x''x - 2y''y + r^2 = 0,$$

$$x(x' - x'') + y(y' - y'') = 0.$$

We may take for the intersections of the orthogonal circles (or of either with the radical axes),

$x_1 = M_1(y' - y'')$, $y_1 = -M_1(x' - x'')$, $x_2 = M_2(y' - y'')$, $y^2 = -M_2(x' - x'')$;

where M_1, M_2 are given by

$$\left[(x' - x'')^2 + (y' - y'')^2\right]M^2 + 2(x'y'' - x''y')M + r^2 = 0 \quad \ldots\ldots \text{(A)}.$$

Let the fixed circle cut the join of centres in the ratio $\lambda : 1$; the values of λ (λ_1, λ_2) are given by

$$\lambda^2(x''^2 + y''^2 - r^2) + 2\lambda(x'x'' + y'y'' - r^2) + (x'^2 + y'^2 - r^2) = 0 \quad \ldots \text{(B)}.$$

The discriminants of (A) and (B) are equal with opposite signs; whence the first part of the question follows.

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The tangents at x_1, y_1 are

$$(x_1-x')\,x+(y_1-y')\,y+(r^2-x_1x'-y_1y')=0,$$
$$(x_1-x'')\,x+(y_1-y'')\,y+(r^2-x_1x''-y_1y'')=0;$$

$$\therefore\quad \cos\theta=(x'x''+y'y''-r^2)/(x'^2+y'^2-r^2)^{\frac{1}{2}}(x''^2+y''^2-r^2)^{\frac{1}{2}}$$
$$=(\lambda_1+\lambda_2)/2\lambda_1^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}=\tfrac{1}{2}(\mu^{\frac{1}{2}}+\mu^{-\frac{1}{2}}),\quad \text{since}\quad \mu=\lambda_2/\lambda_1.$$

11179. (Professor Lancelot.)—On donne, dans un plan, trois droites issues d'un même point et faisant entre elles un angle de 120°; lieu géométrique des points de l'espace tels que le triangle obtenu en joignant les projections de ces points sur les trois côtés ou sur leurs prolongements, ait une aire donnée.

Solution by H. W. Curjel, M.A.; Professor Krishnamacharry; *and others.*

Let P be a point on the locus, and ABC the triangle formed by the feet of the perpendiculars from P on the given straight lines, which meet in O.

Then A, B, C are on the circle on OP as diameter. Hence we have

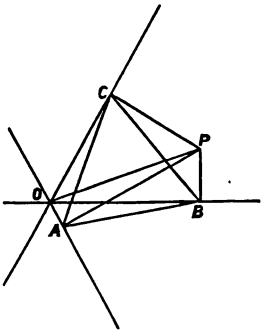
$$\angle CAB=\angle COB,$$

and

$$\angle ACB=\angle BOA.$$

$\therefore$ $\triangle ABC$ is equilateral.

Thus, for its area to be constant, OP is constant. Hence the locus is a circle with centre O.



13077. (Professor Sanjána.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\log(1+\cos x)\,\frac{dx}{\cos x}=\frac{\pi^2}{8}=\frac{3}{2^2}-\frac{5}{6^2}+\frac{11}{30^2}-\frac{13}{42^2}+\frac{19}{90^2}-\dots;$$

$$\int_{\infty}^{0}\cos x\log(x^2)\,dx=\pi;\quad\text{and find}\quad\int^{\infty}\frac{dx}{\left[(1+x^2)(1+mx^2)\right]^{\frac{1}{2}}},\quad\text{where } m>0 \text{ and } <1.$$

Solution by Rev. J. Cullen, S.J.; H. W. Curjel, M.A.; *and others.*

Let

$$u=\int_0^{\frac{1}{2}\pi}\frac{\log(1+\alpha\cos x)}{\cos x}\,dx;$$

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then $\frac{du}{dx} = \frac{2}{(1-\alpha^2)^{\frac{1}{2}}} \tan^{-1}\frac{1-\alpha}{1+\alpha} = -2\frac{d}{d\alpha}\left\{\tan^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1}{2}}\right\}^2$.

$$\therefore \int_0^{\frac{1}{2}\pi} \frac{\log(1+\cos x)}{\cos x}\,dx = -2\int_0^1 \frac{d}{d\alpha}\left\{\tan^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1}{2}}\right\}^2 d\alpha$$

$$= -2\left\{-\frac{\pi^2}{4^2}\right\} = \frac{\pi^2}{8} = 2\left\{\frac{\pi^2}{12} - \frac{1}{4}\frac{\pi^2}{12}\right\}$$

$$= 2\left\{1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \frac{1}{8^2} - \ldots\right\}$$

$$= 2\left\{\frac{3}{2^2} - \frac{5}{6^2} + \frac{11}{30^2} - \frac{13}{42^2} + \frac{19}{90^2} - \ldots\right\}.$$

Again, $\int_\infty^0 \cos x \log(x^2)\,dx = {}_\infty^0\left[2\sin x \log x\right] - 2\int_\infty^0 \frac{\sin x}{x}\,dx$

$$= 2\int_0^\infty \frac{\sin x}{x}\,dx = \pi;$$

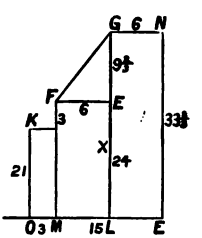
$$\int_0^\infty \frac{dx}{\left[(1+x^2)(1+mx^2)\right]^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{\sec\theta\,d\theta}{(1+m\tan^2\theta)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\left[1-(1-m)\sin^2\theta\right]^{\frac{1}{2}}}$$

$$= \mathrm{F}\left[(1-m)^{\frac{1}{2}}, \tfrac{1}{2}\pi\right].$$

12785. (Professor Davison.)—For the purpose of locating the most eligible point for a county seat, it is required to determine the centre of a county whose dimensions are as follows:—Beginning at the south-west corner, thence east 15 miles, thence north $33\frac{1}{3}$ miles, then west 6 miles to north end of a meridian line running south through the county; thence south-westerly to a point being 6 miles west from the meridian line, and $9\frac{1}{3}$ miles south of its north end, thence south 3 miles, thence west 3 miles, thence south 21 miles to place of beginning.

Solution by Rev. J. L. Kitchin, M.A.; Prof. Radhakrishnan; *and others.*

The solution of this problem is the same thing as find the centre of gravity of a plane figure of uniform density.



$\square GE = 33\frac{1}{3} \times 6 = 202$, ht. of c.g. $\frac{101}{6}$,

$\triangle GEF = 3 \times \frac{29}{3} = 29$, ,, $27\frac{2}{3}$,

$\square FL = 6 \times 24 = 144$, ,, 12,

$\square KM = 21 \times 3 = 63$, ,, $\frac{21}{2}$,

$$\overline{Y}\,(438) = \frac{(101)^2}{3} + 29 \times \frac{245}{9} + 144 \times 12 + \frac{63 \times 21}{2};$$

whence $\overline{Y} = 15{\cdot}02118$, from OE; and *north*,

$\overline{X} = 6{\cdot}814$ from EN, *west*;

and X marks the central point.

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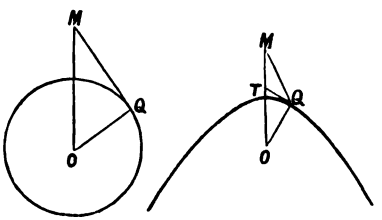
3897. (J. B. Sanders.)—(1) A given weight W is kept at rest on a circular arc by a weight P attached to a cord which passes over a point M in a vertical line through the centre of the circle. Required the position of W, supposing no friction at M. (2) Instead of a circle as in (1), let the curve be a hyperbola with its transverse diameter vertical, the point M being at its centre.

Solution by H. W. Curjel, M.A.; Professor Gopalachanar; *and others.*

(1) Let Q be the position of the weight when in equilibrium, and O the centre of the circle.

Then $W : P = OM : MQ$.

This determines the position of Q.



(2) Let the tangent and normal at Q cut the transverse axis in T and O, and let $\angle OMQ = \theta$.

Then
$$\sin QOM = \frac{a^2 \sin\theta}{(a^4 \sin^2\theta + b^4 \cos^2\theta)^{\frac{1}{2}}},$$

and
$$\sin MQO = \frac{(a^2+b^2)\sin\theta\cos\theta}{(a^4 \sin^2\theta + b^4 \cos^2\theta)^{\frac{1}{2}}};$$

$$\therefore \quad W : P = OM : MQ = (a^2+b^2)\cos\theta : a^2.$$

This determines θ, and hence the position of Q.

12980. (Professor Shields.)—A railroad turn-table 100 feet long is balanced upon a pivot in the centre of a circular track 100 feet in diameter. How far does a man walk who starts at one end of the table and walks, at a uniform rate, the entire length of the table in the same time that the table makes two revolutions, if the table starts to turn at the same time the man starts to walk?

Solution by J. W. West, Professor Radhakrishnan, *and others.*

Equation to curve, $r = (a/2\pi)\,\phi = \rho\phi$. Let S equal total distance.

Then
$$\begin{aligned} S &= \frac{2}{\rho}\int_0^a dr\,(\rho^2+r^2)^{\frac{1}{2}} + A \\ &= \frac{1}{\rho}\left[r(\rho^2+r^2)^{\frac{1}{2}} + \rho^2 \log_e \left\{ \frac{r+(\rho^2+r^2)^{\frac{1}{2}}}{\rho} \right\} \right]_0^a \\ &= a\left[(1+4\pi^2)^{\frac{1}{2}} + \tfrac{1}{2}\pi \log_e \left\{ 2\pi + (1+4\pi^2)^{\frac{1}{2}} \right\} \right] \\ &= 338{\cdot}305 \text{ feet, nearly.} \end{aligned}$$

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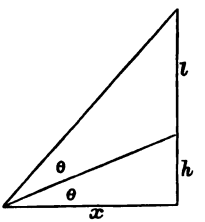
13046. (Professor JULES.)—Une colonne verticale de hauteur h est surmontée d'un mât vertical de hauteur l. On demande à quelle distance horizontale du pied de la colonne il faudra se placer pour voir la colonne et le mât sous des angles égaux.

Solution by W. E HEAL, Prof. KRISHMACHANDRA DE, *and others.*



If x be the distance and θ the angle, we have

$$\tan\theta = \frac{h}{x},$$

$$\tan 2\theta = \frac{h+l}{x} = \frac{2\tan\theta}{1-\tan^2\theta};$$

$$\therefore \quad x = h\left(\frac{l+h}{l-h}\right)^{\frac{1}{2}}.$$

12780. (Professor HUDSON, M.A.)—A unit is divided into three parts so that by taking two of them I may get either $\frac{3}{5}$ or $\frac{9}{10}$ of the unit; what fraction shall I get if I take the third pair. Express each of the parts as a percentage of the whole.

Solution by T. SAVAGE, Professor CHAKRIVARTI, *and others.*

Denoting the parts by $1/x$, $1/y$, $1/z$, we have

$$1/x+1/y+1/z = 1, \quad 1/x+1/y = \tfrac{3}{5}, \quad 1/x+1/z = \tfrac{9}{10}.$$

Whence $1/y+1/z = \frac{1}{2}$; also, $1/x = \frac{1}{2}$, $1/y = \frac{1}{10}$; $1/z = \frac{2}{5}$.

12696. (J. J. BARNIVILLE, B.A.)—Prove that

$$\frac{1}{2.3.4.5.6.7}+\frac{1}{8\ldots13}+\ldots = \frac{\pi}{80\sqrt{3}}+\tfrac{1}{120}-\frac{2\log 2}{45},$$

$$\frac{1}{1.3.5.7.9.11}+\frac{1}{13\ldots23}+\ldots = \frac{23\pi}{23040\,(17+9\sqrt{3})},$$

$$\frac{1}{1.2.3}+\frac{1}{5.6.7}+\frac{1}{9.10.11}+\frac{1}{13.14.15}+\ldots = \tfrac{1}{4}\log 2,$$

$$\frac{1}{1.2.3}\cdot\frac{1}{5.6.7}+\frac{1}{9.10.11}\cdot\frac{1}{13.14.15}+\ldots = \frac{\pi}{480\,(3\sqrt{2}+4)}.$$

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Solution by Rev. T. Roach, M.A.; Professor Radhakrishnan; *and others.*

(α) $S = \frac{1}{120}\left\{\frac{1}{1} - \frac{5}{3} + \frac{10}{5} - \frac{10}{7} + \frac{5}{9} - \frac{1}{11} + \frac{1}{13} - \ldots\right\}$.

$\frac{1}{3}x^2 + \frac{1}{5}x^6 + \ldots = -\frac{1}{8}\log(1+ax).a^4$,

where $a = \cos\frac{1}{3}r\pi \pm i\sin\frac{1}{3}r\pi, \quad r = 0, 1, 2, 3$;

$\frac{1}{3}x^3 + \frac{1}{5}x^9 + \ldots = \frac{1}{8}\log(1+ax).a^3$;

$\frac{1}{7}x^7 + \frac{1}{13}x^{13} = -x + (x + \frac{1}{7}x^7 + \ldots) = -x + \frac{1}{8}\log(1+ax).a^5$.

$\therefore\ S = \frac{1}{120}\left\{x - \frac{1}{8}\log(1+ax)\left[a^4 + 5a^3 + 10a^2 + 10a + 5 + a^5\right]\right\}$

$= \frac{1}{120} - \frac{2\log 2}{45} - \frac{1}{720}\left\{(\sqrt{3})^5(-\frac{1}{3}\pi) - \log(2-1)\right\}$

$= \frac{1}{120} - \frac{2\log 2}{45} + \frac{\pi}{80\sqrt{3}}$.

(β) $S = \frac{1}{3840}\left\{\frac{1}{1} - \frac{5}{3} + \frac{10}{5} - \frac{10}{7} + \frac{5}{9} - \frac{1}{11} + \frac{1}{13} - \ldots\right\}$.

Now $\frac{1}{1}x + \frac{1}{13}x^{13} - \ldots = \frac{1}{13}\Sigma a^{11}\log(1+ax)$,

where $a = \cos\frac{1}{8}r\pi \pm i\sin\frac{1}{8}r\pi$ and $r = 0, 1, 2, \ldots, 6$.

$\frac{1}{3}x^3 + \frac{1}{15}x^{15} - \ldots = \frac{1}{13}\Sigma a^9\log(1+ax\ldots)$.

Hence $S = \frac{1}{3840}\cdot\frac{1}{13}\Sigma\log(1+ax)(a^{11} - 5a^9 + 10a^7 - 10a^5 + 5a^3 - a)$

$= \frac{1}{330240}\cdot\frac{1}{2}\Sigma\log(1+ax).a(a^2-1)^5$.

$\therefore\ S = \frac{1}{330240}\cdot\frac{1}{13}\pi\left\{1^5 - 2(\sqrt{3})^5 + 3(2)^5 - 4(\sqrt{3})^5 + 5.1^5\right\}$

$= \frac{1}{330240}\cdot\frac{1}{13}\pi(102 - 54\sqrt{3}) = \frac{1}{330240}\cdot\frac{1}{2}\pi(17 - 9\sqrt{3})$

$= \frac{23\pi}{23040(17+9\sqrt{3})}$.

(γ) $S = \frac{1}{2}\left\{\frac{1}{1} - \frac{2}{3} + \frac{1}{3} + \frac{1}{5} - \frac{2}{5} + \frac{1}{7} - \ldots\right\}$.

Now $\frac{1}{1}x + \frac{1}{5}x^5 + \ldots = \frac{1}{4}\Sigma a^3\log(1+ax)$,

where $a = \cos\frac{1}{2}r\pi \pm i\sin\frac{1}{2}r\pi$ and $r = 0, 1, 2$.

$\frac{1}{2}x^2 + \frac{1}{6}x^6 + \ldots = \frac{1}{4}\Sigma a^2\log(1+ax)\ldots$.

Hence $S = \frac{1}{4}\cdot\frac{1}{4}\log(1+ax)(a^3 + 2a^2 + a) = \frac{1}{8}\log(1+ax)a(a+1)^2$,

if $r = 2, \Sigma = 0, \quad r = 0, \Sigma = \frac{1}{8}.2^2\log 2$.

Otherwise, $\Sigma = \frac{1}{8}(2\cos\frac{1}{4}r\pi)^2\cos r\pi\log\left\{1 + (\cos\frac{1}{4}r\pi \pm i\sin\frac{1}{4}r\pi)x\right\}$

$= \frac{1}{2}(\cos\frac{1}{4}r\pi)^2\cos r\pi\log 2(1+\cos\frac{1}{2}r\pi) = -\frac{1}{4}\log 2$;

$\therefore\ S = \frac{1}{2}\log 2 - \frac{1}{4}\log 2 = \frac{1}{4}\log 2$.

(δ) $S = \frac{1}{240}\left\{\frac{1}{1} - \frac{4}{3} + \frac{5}{5} - \frac{5}{7} + \frac{4}{9} - \frac{1}{11} + \frac{1}{13} - \ldots\right\}$.

$\frac{1}{2}x^2 + \frac{1}{10}x^{10} + \ldots = -\frac{1}{8}\Sigma a^6\log(1+ax)\ldots$. Hence

$S = \frac{1}{240}\cdot\frac{1}{8}.r^3(1+\cos\frac{1}{4}r\pi)^2\sin\frac{1}{2}r\pi\,(\pm i\cos r\pi)\log\left\{1 + \left[\cos\frac{1}{4}r\pi \pm \sin(\frac{1}{4}r\pi)x\right]\right\}$,

if $r = 0$ or 4, $S = 0$.

$\therefore\ S = \frac{1}{240}(1+\cos\frac{1}{4}r\pi)^2\sin\frac{1}{4}r\pi\cos r\pi(-2)\frac{1}{8}r\pi$

$= \frac{1}{240}\cdot\frac{1}{4}\pi\left\{\frac{1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{2}}\right)^2 - 2(1)^2 + 3.\frac{1}{\sqrt{2}}\left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$

$= \frac{1}{240}\cdot\frac{1}{4}\pi(3\sqrt{2}-4) = \frac{\pi}{480(3\sqrt{2}+4)}$.

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13041. (Professor WOLSTENHOLME.)—If $u = 0$ be the rational equation of the second degree (coordinates rectangular), and if the equation

$$\frac{d^2u^{\frac{1}{3}}}{dx^2} + \frac{d^2u^{\frac{1}{3}}}{dy^2} + \frac{d^2u^{\frac{1}{3}}}{dz^2} = 0$$

represent a paraboloid (a true paraboloid excluding cylinders), prove that it will be a paraboloid of revolution. [It would be interesting to see this proved directly from the general equation, where

$$u \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2a''x + 2b''y + 2c''z + d;$$

viz., if the equation

$$(ax + hy + gz + a'')^2 + (hx + by + fz + b'')^2 + (gx + fy + cz + c'')^2 = (a + b + c)\,u$$

represent a paraboloid, it must be one of revolution.]

Solution by W. E. HEAL, Professor KRISHNAMACHARRY, *and others.*

The discriminant of the equation

(1) $(ax + hy + gz + a'')^2 + (hx + by + fz + b'')^2 + (gx + fy + cz + c'')^2 = (a + b + c)\,u$

is

$$\begin{vmatrix} a^2+h^2+g^2, & ah+bh+fg, & ag+fh+cg, & aa''+b''h+c''g \\ ah+bh+fg, & h^2+b^2+f^2, & gh+bf+cf, & a''h+bb''+c''f \\ ag+fh+cg, & gh+bf+cf, & g^2+f^2+c^2, & a''g+b''f+cc'' \\ aa''+b''h+c''g, & a''h+bb''+c''f, & a''g+b''f+cc'', & a''^2+b''^2+c''^2 \end{vmatrix}$$

$$-(a+b+c)^4 \begin{vmatrix} a, & h, & g, & a'' \\ h, & b, & f, & b'' \\ g, & f, & c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix}.$$

The first determinant vanishes, since it is the product of

$$\begin{vmatrix} a, & h, & g, & a'' \\ h, & b, & f, & b'' \\ g, & f, & c, & c'' \\ a'', & b'', & c'', & d \end{vmatrix} = \mathrm{D}, \text{ by } \begin{vmatrix} a, & h, & g, & 0 \\ h, & b, & f, & 0 \\ g, & f, & c, & 0 \\ a'', & b'', & c'', & 0 \end{vmatrix} = 0;$$

$$\therefore \quad \mathrm{D} = 0, \quad \text{or} \quad (a+b+c) = 0.$$

$\mathrm{D} = 0$ is the condition that $u = 0$ should represent a paraboloid.

If $(a+b+c) = 0$, since the coefficient of z^2 in (1) may be made to vanish, we have $g^2 + f^2 + c^2 = 0$;

$\therefore \quad g = f = c = 0$; or $u = 0$ is again a paraboloid.

We may, then, take u in the form $u \equiv ax^2 + by^2 + 2c''z = 0$.

Equation (1) then becomes $ab\,(x^2+y^2) + 2c''\,(a+b)\,z - c''^2 = 0$, which evidently represents a paraboloid of revolution.

13149. (R. KNOWLES, B.A.)—CP, CD are equi-conjugate radii of an ellipse; tangents at P and D intersect in T; prove that the radii of the

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in-circle and circum-circle of the triangle TPD are respectively

$$2^{-\frac{1}{2}}ab\left[(a^2+b^2)^{\frac{1}{2}}+a\right]^{-1} \text{ and } (a^2+b^2) \div 2^{\frac{3}{2}}b;$$

a and b being the semi-axes of the ellipse.

Solution by Rev. T. Wiggins, S.J.; H. W. Curjel, M.A.; *and others.*

Area of $\triangle PTD = \frac{1}{8}$ of parallelogram formed by tangents parallel to CP, CD $= \frac{1}{2}ab$.

Sides of triangle are $a\sqrt{2}$, $\left[\frac{1}{2}(a^2+b^2)\right]^{\frac{1}{2}}$, $\left[\frac{1}{2}(a^2+b^2)\right]^{\frac{1}{2}}$.

$$\therefore \text{ radius of in-circle} = \frac{ab}{2}\frac{\sqrt{2}}{a+(a^2+b^2)^{\frac{1}{2}}} = 2^{-\frac{1}{2}}ab\left[(a^2+b^2)^{\frac{1}{2}}+a\right]^{-1},$$

$$\text{and radius of circum-circle} = \frac{a\sqrt{2}\,(a^2+b^2)}{4ab} = (a^2+b^2) \div 2^3 b.$$

13047. (Professor Scheffer.)—The floor of a vault forms a square, and all sections parallel to it are squares. The two vertical sections through the middle points of the opposite sides of the floor are equal semi-circles. Find the convex surface and the volume of the vault.

Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

$$\text{Convex surface} = 4\int_0^{\frac{1}{2}\pi} 2a\cos\theta\, a\, d\theta = 8a^2.$$

$$\text{Volume} = \int_0^{\frac{1}{2}\pi}(2a\cos\theta)^2\, a\cos\theta\, d\theta = 4a^3\int_0^{\frac{1}{2}\pi}(1-\sin^2\theta)\cos\theta\, d\theta = \tfrac{8}{3}a^3.$$

13128. (M. Brierley.)—Find the sum, to infinity, of

$$\left\{\frac{1}{3}+\frac{1}{2.3}\right\}-\left\{\frac{1}{5}+\frac{1}{2.5}+\frac{1}{3.5}+\frac{1}{4.5}\right\}$$
$$+\left\{\frac{1}{7}+\frac{1}{2.7}+\frac{1}{3.7}+\frac{1}{4.7}+\frac{1}{5.7}+\frac{1}{6.7}\right\}-\text{ \&c.}$$

Solution by C. W. Bourne, M.A.; Professor Beyens; *and others.*

Let u_3, u_5, ... denote the terms of the given series.

By the differential calculus, it is easily shown that

$$\tan^{-1}x \,.\, \log(1+x^2)^{\frac{1}{2}} = u_3x^3 - u_5x^5 + u_7x^7 - \ldots.$$

Put $x = 1$, and the sum of the given series becomes $\frac{1}{4}\pi \,.\, \log 2^{\frac{1}{2}}$.

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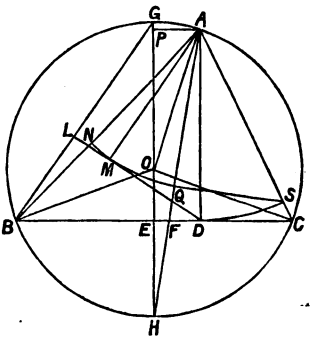
13100 & 13151. (M. Brierley.)—Given the difference of the segments of the base made by the perpendicular from the vertex, the difference of the angles at the base and the rectangle under the sum of the sides and the line bisecting the vertical angle, to construct the triangle.

Solution by D. Biddle, Professor Krishnamacharry, *and others.*



Let AOG = the given difference of the angles at the base, and AP = half the given difference of the segments. Then O may be taken as the centre and OA as the radius of the circumcircle. Produce GO to H, so that GH may be a diameter perpendicular to AP, and join AH. Then AF is the line bisecting the vertical angle, but F is not yet found. In AH, take AQ so that AQ . AO = one-fourth of the given rectangle, AF (AB + AC). Draw QS at right angles to AH, so that the perpendicular to GH from C on AS produced, and also on the circumcircle, shall be tangential to the circle of radius AS and centre A.

The angle SAH ($=\theta = \frac{1}{2}$A) can be found from the following cubic, in which α = AHG = $\frac{1}{2}$AOG; AO, = R, being taken as unity:

$$\cos^3\theta - \sin^2\alpha \cdot \cos\theta - \tfrac{1}{2}AQ = 0.$$

This can be effected by the method of conic geometry set forth in the solution of Quest. 12169 (Vol. lxi., pp. 27, 28).

Now, $AB = 2\cos(\theta - \alpha)$ and $AC = 2\cos(\theta + \alpha)$;

$$\therefore\ AB + AC = 4\cos\alpha\cos\theta = AH \cdot GB.$$

Moreover, $AD : AF = AH : GH$;

$$\therefore\ AF(AB + AC) = AD \cdot GB \cdot GH = 4AQ.$$

But $AD = \cos 2\alpha + \cos 2\theta = 2(\cos^2\theta - \sin^2\alpha)$, and $GB = 2\cos\theta$, whilst $GH = 2$; $\therefore\ AQ = 2\cos\theta(\cos^2\theta - \sin^2\alpha)$;

whence the above cubic.

It is also interesting to note that the perpendicular from D on GB is tangential to the circle of centre A and radius AQ.

13133. (Professor Sanjána.)—Let the sides AB, CB of a triangle be produced to B_1, B_2, respectively, and let lines parallel to the symmedian of B be placed in the angles ABB_2, CBB_1; call the locus of their mid-points the ex-symmedian of B; proceed similarly for the other angles. Then (1) prove that the ex-symmedians of two angles and the symmedian

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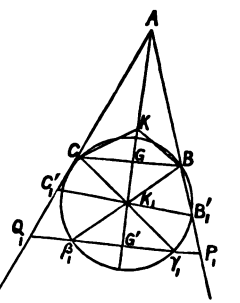
of the third meet in a point; call the three points thus obtained K_1, K_2, K_3, the symmedian point being K; (2) show that the circle described with K_1 as centre and K_1B or K_1C as radius cuts two sides produced at the extremities of a diameter; so for K_2 and K_3; (3) if BK_1, CK_1 meet this circle again in β_1, γ_1, and AB, AC meet $\beta_1\gamma_1$ produced in P_1, Q_1, find the circum-radius R_1 of AP_1Q_1, also of BP_2Q_2, CP_3Q_3; (4) prove that $R_1^{-1} + R_2^{-1} + R_3^{-1} = R^{-1}$; and (5) prove that the distance

$$KK_1 = 2a^2bcm_1/(b^2 + c^2 - a^2 + 2bc),$$

where m_1 is the median corresponding to A; also prove similar expressions for KK_2, KK_3.

Solution by W. E. Jeffares, B.A.; H. W. Curjel, M.A.; *and others.*



The ex-symmedian of B bisects all lines parallel to BK, and terminated by BA, BC; therefore the symmedian of B bisects all lines parallel to the ex-symmedian of B and terminated by BA, BC; therefore the ex-symmedian of B is the tangent at B to the circumcircle. Also K_1, where it meets AK, is the harmonic conjugate of K with respect to A and G, where AK meets BC; therefore ex-symmedian of C also passes through K_1, and $BK_1 = CK_1$.

Let circle with centre K_1 and radius K_1B cut AB, AC in B_1, C_1. Then B_1C_1 is anti-parallel to BC, and is therefore bisected by AK; but AK is not perpendicular to B_1C_1.

Therefore B_1C_1 passes through K_1. Let AK cut P_1Q_1 in G'; then

$$K_1G = K_1G' \quad \text{and} \quad K_1G : K_1A = KG : KA = a^2 : b^2 + c^2;$$

$$\therefore \quad \frac{AG'}{AG} = \frac{a^2 + b^2 + c^2}{b^2 + c^2 - a^2}; \quad \therefore \quad R_1 = \frac{a^2 + b^2 + c^2}{b^2 + c^2 - a^2} R.$$

Similarly, $$R_2 = \frac{a^2 + b^2 + c^2}{a^2 - b^2 + c^2} R, \qquad R_2 = \frac{a^2 + b^2 + c^2}{a^2 + b^2 - c^2} R.$$

$$\therefore \quad R_1^{-1} + R_2^{-1} + R_3^{-1} = R^{-1}.$$

Again, $$B_1C_1 = \frac{BC}{\cos A}; \quad \therefore \quad AK_1 = \frac{m_1}{\cos A}.$$

Also $$AK = \frac{b^2 + c^2}{a^2 + b^2 + c^2} AG, \quad AK_1 = \frac{b^2 + c^2}{b^2 + c^2 - a^2} AG;$$

$$\therefore \quad KK_1 = \frac{2(b^2 + c^2) a^2 AG}{(a^2 + b^2 + c^2)(b^2 + c^2 - a^2)}$$

$$= \frac{2a^2m_1}{\cos A (a^2 + b^2 + c^2)} = \frac{4a^2bcm_1}{b^4 + c^4 - a^4 + 2b^2c^2}.$$

Similarly, $$KK_2 = \frac{4ab^2cm_2}{a^4 + c^4 - b^4 + 2a^2c^2} \text{ and } KK_3 = \frac{4abc^2m_3}{a^4 + b^4 - c^4 + 2a^2c^2}.$$

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11707 & 11720. (A. E. Jolliffe, M.A.)—From two points B, C. tangents are drawn to a conic S, and these four tangents, together with the polars of B and C with respect to S, all touch a conic α. Similarly the pairs of points CA, AB determine the conics β and γ, respectively. Prove, by pure geometry, that, if A lie on α, then B lies on β, and C on γ.

(11720.) In Quest. 11707, prove also that the sides of the triangle ABC meet the conic S in three pairs of points, such that the three pairs of straight lines joining them to the opposite vertices intersect by threes in two points.

Solution by H. W. Curjel, M.A.; Professor Coondoo; *and others.*

11707. Project the points of contact of tangents from A to S into the circles. Then S becomes a circle, centre A (Fig. 1), and β and γ become parabolæ with common focus A and vertices V, W on AB, AC such that $AV.AB = AW.AC = (\text{radius of S})^2$.

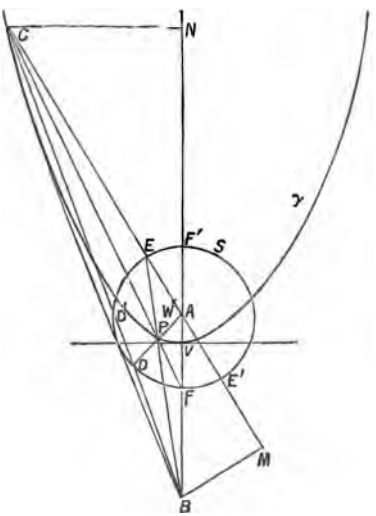

Fig. 1.

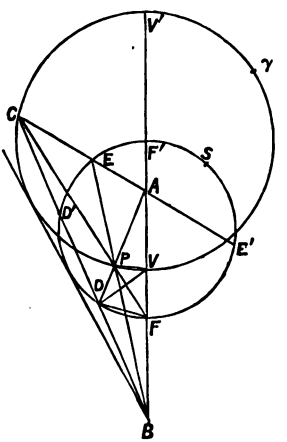

Fig. 2.

Draw CN, BM perpendicular to BA, CA. Then

$$\frac{BM}{CN} = \frac{AM}{AN} = \frac{AB}{AC} = \frac{AW}{AV} = \frac{WM}{VN}; \quad \therefore \quad \frac{BM^2}{CN^2} = \frac{4AW.WM}{4AV.VN};$$

therefore, if C is on γ, B is on β. Similarly, by projecting the figure so that the points of contact of tangents from B to S become the circules, we can show that A is on α.

11720. The circle S and the parabola γ are in a quadrilateral of which B and A are opposite vertices. Therefore we can project them into two

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circles, of which B and A are centres of similitude (Fig. 2). In this figure, γ t uches the chord of contact of tangents from B to S; therefore S passes through the centre of circle γ.

Let BA meet S in FF′ and γ in VV′, and let BC cut S in DD′, D′F′ being parallel to CV′; then CV′FD are concyclic. Since the range (FF′, BV) is harmonic, DF bisects $\angle$ BDV. Let AD cut γ in P, PV being not parallel to DF; then DFVP are concyclic.

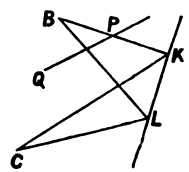
$$\therefore \quad \angle FPV = \angle FDV = \angle BDF = CV'V.$$

Therefore FPC are collinear, *i.e.*, P being the point where FC meets γ, BC, AP cut on S at D. Similarly BP, AC cut S in E. Let CA cut S again in E′; then it may be shown in the same way AD′, BE′, CF′ cut in a point Q on γ. Hence, also, in the original figure, if S cuts BC, CA, AB in D, D′; E, E′; F, F′; AD, BE, CF cut in a point P, and AD′, BE′, CF′ cut in a point Q, and P, Q lie on γ; and in the same way it may be shown that P, Q lie on α, β.

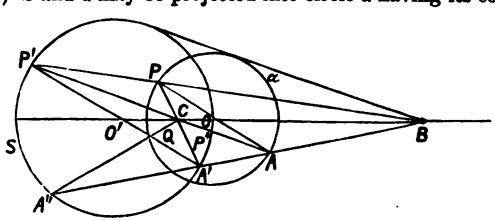
Therefore P, Q are points common to α, β, γ.

[The Proposer gives these proofs of his theorems:—

(11707.) β is the envelope of lines joining points conjugate with regard to S on the polars of A and C. α', the reciprocal of α with respect to S, is the locus of points such that the lines joining them to B and C are conjugate with respect to S. Let the polar of A with respect to S meet α' in K and L. Let the polar of C meet BK and BL in P and Q. Then P is the pole of CK, and Q is the pole of CL. Therefore P, K are conjugate points, as are Q, L. Therefore BK and BL are the tangents from B to β. So CK and CL are the tangents from C to γ. If A is on α, the polar of A touches α'; therefore K and L coincide, and therefore B is on β and C on γ.



(11720.) S and α may be projected into circle α having its centre on S.



B and C are the centres of similitude. Let B be the external centre. Let O, O′ be the centres of α and S. Let AO cut α in P. Let BA meet S in A′ such that O′A′ is parallel to OA. Let O′A′ meet S again in P′. Then B, P, P′ are collinear, O, O′ bisect PA, P′A′, which are parallel; therefore PA′, P′A cut on OO′ in the harmonic conjugate of B with respect to OO′, that is to say, in C. Hence three of the lines joining A, B, C to the points where S meets the sides of ABC are concurrent. So therefore are the other three. One of the points of concurrency is on α. In the original figure there will be no distinction between the two

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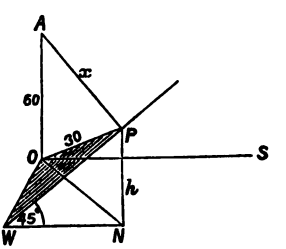
points; therefore the other will be on α (and obviously on β and γ); or we may prove it as follows with the same figure.

Let BA meet S again in A″, and let CA″ meet α in Q, and CP′ meet S in P″, such that A″, Q, P″, A are concyclic. BP″ will meet α again in a point concyclic with A, A″, P″, that is to say, in Q.]

13004. (Professor CHAKRIVARTI.)—A tower, whose height is 60 feet, is situated on the side of a hill which rises due south at an angle of 45°. What length of ladder will be necessary to scale the tower, supposing the foot of the ladder to be placed at a distance of 30 feet in a south-west direction from the base of the tower?

Solution by W. J. DOBBS, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let OS, OW be straight lines drawn due south and west, respectively, through the base of the tower OA. Let P be the foot of the ladder. Let the vertical through P meet the plane SOW in N. Then ON bisects angle SOW. Let the vertical plane through P, which runs due north and south, meet OW in W. Then WP is a line of greatest slope of the hill, and makes 45° with WN, which is parallel to OS.



If PN be of length h feet, we have

$$WN = h \text{ feet}, \quad ON = h\sqrt{2} \text{ feet}.$$

Hence, if x feet be the length of the ladder,

$$x^2 = (h\sqrt{2})^2 + (60-h)^2 = 60^2 - 120h + 3h^2.$$

Also $30^2 = h^2 + (h\sqrt{2})^2$; $\therefore\ h = 10\sqrt{3}$;

$\therefore\ x^2 = 60^2 - 1200\sqrt{3} + 30^2 = 300\,(15 - 4\sqrt{3}) = 2421{\cdot}6$; $\therefore\ x = 49{\cdot}2$;

$\therefore$ ladder is of length 49·2 feet.

13148. (W. C. STANHAM.)—P and Q are the points of contact of the tangents from (x', y') to the ellipse $x^2/a^2 + y^2/b^2 = 1$. Prove that, if R be a point on the ellipse such that its eccentric angle ϕ satisfies the relation $x'^2 \sec^2\phi - y'^2 \operatorname{cosec}^2\phi = a^2 - b^2$, with the further condition that

$$ay' \operatorname{cosec}\phi - bx' \sec\phi \quad \text{and} \quad ax' \sec\phi - by' \operatorname{cosec}\phi + a^2 - b^2$$

are of the same sign, the difference of the arcs PR and QR (measured in opposite directions round the ellipse) is either zero or equal to the length of a known straight line.

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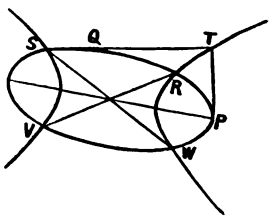
Solution by the PROPOSER, Professor KRISHMACHANDRA DE, *and others.*

Let a', b' be the semi-axes of the ellipse (confocal) through T (x', y'), and let α be the eccentric angle of T.



Then the hyperbola through T is

$$x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = a'^2 - b'^2 = a^2 - b^2.$$

R will be a point on this hyperbola if

$$\sec^2 \phi = \sec^2 \alpha \quad \text{and} \quad \operatorname{cosec}^2 \phi = \operatorname{cosec}^2 \alpha,$$

i.e., if $\phi = \alpha,\ \pi - \alpha,\ \pi + \alpha,$ or $2\pi - \alpha$.

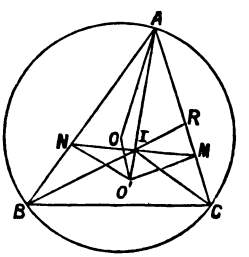
The condition of sign leads to the rejection of the second and fourth of these values for ϕ, and MACCULLAGH'S theorem gives the required relation for the first and third (the points R and V in the figure). It is clear that no value of ϕ besides the four mentioned above will satisfy

$$x'^2 \sec^2 \phi - y'^2 \operatorname{cosec}^2 \phi = a'^2 - b'^2.$$

13126. (Rev. G. RICHARDSON, M.A.)—If a circle be drawn touching two sides of a triangle so that the chord of contact passes through the incentre or one of the excentres, prove that the circle touches the circumcircle.

Solution by Professors SANJÁNA, MUKHOPADHYAY, *and others.*

This is the last part of Quest. 12424 (by EDITOR); see p. 59 of Vol. LXII. The following is a proof by elementary trigonometry:—



Let O and I be the circumcentre and incentre of a triangle ABC; draw IM at right angles to AI to meet AC, and MO′ at right angles to AM to meet AI produced. Then O′ is the centre of the circle touching AC and AB, and having the chord of contact MN passing through I.

The radius of this circle O′M

$$= r \sec^2 \tfrac{1}{2}A.$$

Also $OA = R$, $O'A = r \sec^2 \frac{1}{2}A \operatorname{cosec} \frac{1}{2}A$, $\angle OAO' = \frac{1}{2}(C - B)$;

hence $OO'^2 = R^2 + r^2 \sec^4 \frac{1}{2}A \operatorname{cosec}^2 \frac{1}{2}A - 2Rr \sec^2 \frac{1}{2}A \operatorname{cosec} \frac{1}{2}A \cos \frac{1}{2}(C - B)$

$$= R^2 + r^2 \sec^4 \tfrac{1}{2}A + r^2 \sec^2 \tfrac{1}{2}A \operatorname{cosec}^2 \tfrac{1}{2}A - 2Rr \sec^2 \tfrac{1}{2}A \operatorname{cosec} \tfrac{1}{2}A \cos \tfrac{1}{2}(C - B)$$

$$= R^2 + r^2 \sec^4 \tfrac{1}{2}A - 2Rr \sec^2 \tfrac{1}{2}A \operatorname{cosec} \tfrac{1}{2}A \left\{\cos \tfrac{1}{2}(C - B) - 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C\right\}$$

$$= R^2 + r^2 \sec^4 \tfrac{1}{2}A - 2Rr \sec^2 \tfrac{1}{2}A \operatorname{cosec} \tfrac{1}{2}A \cos \tfrac{1}{2}(C + B)$$

$$= (R - r \sec^2 \tfrac{1}{2}A)^2.$$

$\therefore$ $OO' = R - r \sec^2 \frac{1}{2}A$ = difference of the radii of the two circles; hence the proposition for the incircle. So also for the excircles.

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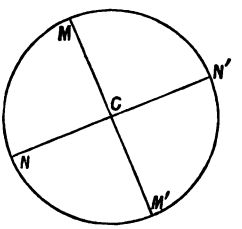
13136. (Professor BARISIEN.)—Le cercle de courbure C en un point quelconque M d'une ellipse rencontre la normale en M en un second point M'. Le diamètre du cercle C perpendiculaire à MN' rencontre ce cercle en N et N'. Montrer que chacune des courbes lieux des points M', N, N' a une aire équivalente à celle de l'ellipse.

Solution by H. W. CURJEL, M.A.; W. E. JEFFARES, B.A.; *and others.*



The theorem is true for any closed curve described by M of which the evolute is a closed curve. Since C is the centre of curvature of the curve (M), the tangents to the curves (N), (N') at N, N' are both perpendicular to NCN'.

Therefore, by the first part of Quest. 12796 (Vol. LXIV., p. 63), (N) and (N') are equal in area.

Again, by Professor ELLIOTT's extension of HOLDITCH's theorem [since the area described by radii vectores drawn from a fixed point equal and parallel to the diameters of curvature of (M)

$$= 4\{(M) - (\text{evolute of } M)\}],$$

$$2(C) = M + M' - \tfrac{4}{3}\{(M) - (C)\}; \quad \therefore \quad M = M';$$

and

$$2(C) = N + N' - \tfrac{4}{3}\{(M) - (C)\};$$

$$\therefore \quad (N) + (N') = 2(M); \quad \therefore \quad (N) = (N') = (M) = (M').$$

13141. (Professor MYERS.)—A flexible cord of given length is suspended from two points whose coordinates are (x, y) and (x', y'); find what the condition of the chord must be in order that it may hang in the arc of a circle.

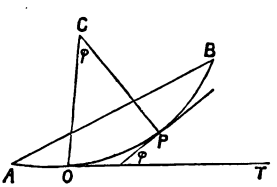
Solution by Professor KRISHNAMACHARRY, W. C. STANHAM, *and others.*

Since the chord AB and the arc AB are each of given length, the circle in which the cord hangs is completely determined.

Let C be its centre, O the lowest point of the string, OT the tangent at O;

$$CO = a.$$

Then equation of circle is $s = a\phi$.



And, if m is the mass per unit length at the point (x, y),

$$m\frac{ds}{dx} \propto \frac{d^2y}{dx^2}, \quad \text{and} \quad \frac{dy}{dx} = \tan\phi;$$

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$\therefore \quad \frac{d^2y}{dx^2} = \sec^2\phi \frac{d\phi}{ds}\frac{ds}{dx}, \quad \frac{ds}{d\phi} = a, \quad \frac{ds}{dx} = \sec\phi; \qquad \therefore \quad \frac{d^2y}{dx^2} = \sec^3\phi/a;$

$\therefore \quad m \propto \sec^3\phi$ or $\sec^3 s/a$, where s is measured from a known point O of the cord.

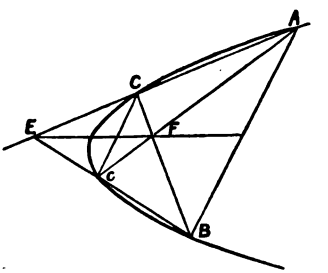
13116. (R. F. Davis, M.A.)—If A, B, C, D be four points on a parabola, the portions of the diameter bisecting AB, intercepted respectively by AC, BD and AD, BC, will be equal.

Solution by Professor Sanjána; H. W. Curjel, M.A.; *and others.*

Draw chord Cc parallel to AB. Let AC, cB and Ac, CB cut in E and F. Then EF is the diameter bisecting AB, and F is on the polar of E; therefore EF is bisected by the parabola.

Similarly, if AD, BD cut EF in G and H, GH is bisected by the parabola;

$\therefore$ EH = GF.



13144. (Editor.)—If ABC be a triangle in a circle, AD the bisector of the angle A, AP a tangent meeting BC at P, PQR a perpendicular to AD meeting the circle in Q, R, prove that

$$AQ^2 = BQ \cdot CQ, \quad AR^2 = BR \cdot CR.$$

Solution by H. W. Curjel, M.A.; R. F. Davis, M.A.; *and others.*

Let AD cut the circumcircle again in E.

Then $\angle PAD = \angle ACE$
$= \angle ADP$;

$\therefore$ PQR bisects AD.

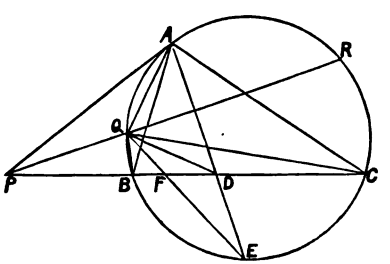
Let QE cut BC in F.

$\angle QDF = \angle QAP = \angle QED$;

$\therefore$ QD touches circle DEF;

$\therefore \quad AQ^2 = QE \cdot QF$
$= QB \cdot QC$

(for $\triangle$s BQE, FQC are similar). Similarly, $AR^2 = RB \cdot RC$.



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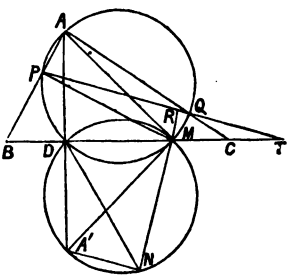
13131. (Professor NEUBERG.)—On donne un triangle ABC. Par un point quelconque M de BC on mène MP perpendiculaire à AB, MQ perpendiculaire à AC, et l'on trace PQ. Démontrer que la perpendiculaire abaissée de M sur PQ enveloppe une parabole.

Solution by H. W. CURJEL, M.A.; Professor SANJÁNA; *and others.*



Draw AD perpendicular to BC, and produce AD to A′ so that DA′ = AD. On AM, A′M as diameters describe circles ADM, A′DM. The former passes through P and Q.

Let MR (the perpendicular from M on PQ) cut circle A′DM in N, and let PQ, BC cut in T. Let

$$\angle MAC = \theta = \angle MPQ.$$

Then
$$\angle A'MD = \angle AMD = C + \theta$$
$$\angle DMN = \angle RMC = \tfrac{1}{2}\pi - RTM$$

and
$$= \tfrac{1}{2}\pi - (\tfrac{1}{2}\pi - B - \theta)$$
$$= B + \theta;$$

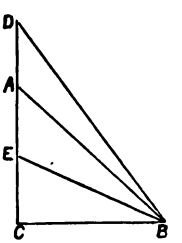
$$\therefore \quad \angle A'DN = \angle A'MN = B - C.$$

Therefore DN is a fixed straight line, and A′NM is a right angle.

Therefore the envelope of RMN is a parabola with focus A′ and tangent at vertex DN.

13042. (Professor SANJÁNA.)—In an isosceles triangle ABC, right-angled at C, CA is produced to D, so that CD = AB; DB is joined, and the angle DBE is made = BDC. Prove that CE is one-third of DE. [This is the geometrical proof of $\tan^{-1}\sqrt{2} - \cot^{-1}\sqrt{2} = \sin^{-1}\tfrac{1}{3}$.]

Solution by Rev. T. WIGGINS, B.A.; Professor LAMPE; *and others.*



$$CD^2 = AB^2 = 2AC^2 = 2BE^2 - 2CE^2;$$
$$\therefore \quad DE^2 + CE^2 + 2CE \cdot DE = 2DE^2 - 2CE^2;$$
$$\therefore \quad 3CE^2 + 2CE \cdot DE = DE^2;$$
$$\therefore \quad 3(CE^2 + CE \cdot DE) = DE(CE + DE),$$
$$\text{or} \quad 3CE(CE + DE) = DE(CE + DE),$$
$$\text{or} \quad DE = 3CE.$$
$$\angle EBC = \angle DBC - \angle DBE$$
$$= \angle DBC - \angle CDB;$$
$$\therefore \quad \sin^{-1}\tfrac{1}{3} = \tan^{-1}\sqrt{2} - \cot^{-1}\sqrt{2}.$$

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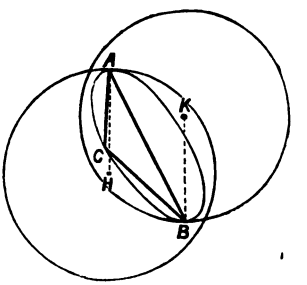
12598. (W. J. Dobbs, M.A.)—It is desired to place a given triangle in such a position that the times of descent from rest down its three sides, considered as smooth material straight lines, may be equal. Show that this cannot be done unless the triangle is right-angled; and that, if this condition be satisfied, the hypotenuse of the triangle may be placed in any position, and the plane of the triangle must then be turned about its hypotenuse until it is at right angles to the vertical plane which contains the hypotenuse.

Solution by the Proposer, Professor Mukhopadhyay, *and others.*

In order that the times down AB and AC may be equal, it is necessary and sufficient that B and C should be points on a *sphere* of which A is the *highest* point.

In order that the times down AB and CB may be equal, it is necessary and sufficient that A and C should be points on a *sphere* of which B is the *lowest* point.



Thus, for the times down AB, AC, CB to be equal, C must be a point on the circular section of two spheres of which A and B are the tangent and lowest points respectively.

Hence, take any sphere centre H and highest point A. Take *any other point* B on the sphere, and describe a sphere (centre K) passing through A and having B for its lowest point. Only one such sphere can be described.

Now take *any point* C on the circular section of the two spheres. Then the triangle ABC satisfies the conditions of the problem.

Now AB is a diameter of the circular section Therefore the angle C must be a right angle. This is the only limitation upon the *shape* of the triangle. Also, as B is *any* point on the first sphere, the line AB may be in any position. Again, the plane of the triangle is that of the circular section of the two spheres, and is therefore perpendicular to the line HK, and therefore to the plane AHBK, which is the vertical plane containing AB.

Thus the triangle must be right-angled, its hypotenuse may be placed *in any position*, and the plane of the triangle must then be turned about its hypotenuse until it is at right angles to the vertical plane containing the hypotenuse.

[N.B.—The question may be otherwise expressed:—The well-known property of the times of descent from rest down chords of a *vertical* circle drawn through either the highest or lowest points of the circle may be extended to the case in which the plane of the circle is *not vertical*.]

13147. (R. F. Davis, M.A.)—ABCD is a quadrilateral inscribed in a circle centre O. If, upon AB, CD, points P, S be taken, respectively,

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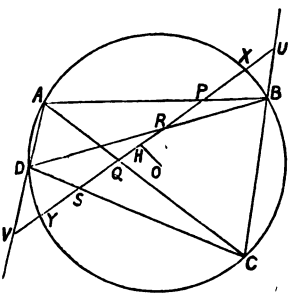
such that OP = OS, and PS produced meet the circle in X, Y. and AC, BC, AD, BD in Q, U, V, K, respectively, prove that UX = VY, PX = SY, and PQ = RS.

Solution by R. F. DOBBS, M.A.; H. W. CURJEL, M.A.; *and others.*

The circle and the line pairs AC, BD; BC, AD; AB, DC cut PS in involution; and, since PS and XY are both bisected at H (the foot of perpendicular from O on PS), the common harmonic conjugates of the pairs R, Q; P, S; X, Y; U, V are H and the point at infinity on PS.



$\therefore$ H bisects RQ, PS, XY, UV.

$\therefore$ UX = VY,
PX = SY,
PQ = RS.

13150. (Rev. G. RICHARDSON, M.A.)—The locus of the orthocentre of the triangle formed by three normals to a parabola, two of which are fixed, is another parabola.

Solution by H. W. CURJEL, M.A.; Professor CHAKRAVARTI; *and others.*

Let the fixed normals be

$$m^3y + m^2x - 2m^2a - a = 0, \quad n^3y + n^2x - 2an^2 - a = 0 \quad \ldots\ldots(1, 2),$$

and the other normal be $\theta^3y + \theta^2x - 2a\theta^2 - a = 0$ .....(3).

At the intersection of (1) and (3)

$$y = -\frac{(m+\theta)a}{m^2\theta^2}, \quad x = a\frac{\theta^2(2m^2+1) + m\theta + m^2}{m^2\theta^2};$$

therefore orthocentre is on $nx - y = a\dfrac{\theta^2(2m^2n+n) + mn\theta + m^2n + m + \theta}{m^2\theta^2}$;

similarly it is on $mx - y = a\dfrac{\theta^2(2n^2m+m) + mn\theta + n^2m + n + \theta}{n^2\theta^2}$;

therefore at orthocentre $y = a\dfrac{m+n}{mn} + \dfrac{a(mn+1)}{mn\theta}$,

$$x = a\frac{2m^2n^2 + mn + m^2 + n^2}{m^2n^2} + \frac{a(m+n)(mn+1)}{m^2n^2\theta} + \frac{a(mn+1)}{mn\theta^2}$$

= a quadratic function of y.

Therefore locus of orthocentre is a parabola with axis parallel to the axis of the given parabola.

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9809. (W. J. C. SHARP.)—Given forces act along the sides of a triangle, in the same sense. The value of the mean sum of their moments about a random point in the triangle is the mean of their moments about the vertices of the triangle. If the forces be such that if, applied at a point, there would be equilibrium, the sum of the moments is the same for all points.

Solution by H. J. WOODALL, A.R.C.S.; Prof. MUKHOPADHYAY; *and others.*

Let the given forces be p, q, r; then their moment

$$M = p\alpha + q\beta + r\gamma = \alpha\,(p - ar/c) + \beta\,(q - br/c) + 2\Delta r/c$$

(since $a\alpha + b\beta + c\gamma = 2\Delta$).

The mean of this (with α constant) is

$$M = \left\{\alpha\,(bcp - \tfrac{1}{3}abr - \tfrac{1}{3}acq) + \Delta\,(br + cq)\right\}/bc.$$

The mean of this all over the triangular area is

$M = \frac{2}{3}\Delta\,(p/a + q/b + r/c)$ = mean of their moments about the vertices.

In the second case p, q, r are proportional to a, b, c respectively; therefore $M = p\alpha + q\beta + r\gamma = p\,(a\alpha + b\beta + c\gamma)/a$

$= 2p\Delta/a$ = constant for all points in or out of the triangle.

9926. (Capitaine DE ROCQUIGNY.)—Je dispose neuf nombres entiers consécutifs $n-4, n-3, \ldots, n+4$, de manière que le nombre moyen n occupe le centre d'un échiquier de neuf cases, et que les autres nombres se succèdent dans l'ordre de grandeur croissante suivant la loi du cavalier; je forme ainsi le *déterminant du cavalier*. Comment faut-il choisir n pour que la valeur du déterminant soit un carré parfait?

Solution by H. J. WOODALL, A.R.C.S.; Prof. CHAKRIVARTI; *and others.*

If the determinant is

$$\begin{vmatrix} n+2, & n-3, & n+4 \\ n-1, & n, & n+1 \\ n-4, & n+3, & n-2 \end{vmatrix} = 0,$$

or if

$$\begin{vmatrix} n+3, & n-3, & n+1 \\ n-1, & n, & n+4 \\ n-4, & n+2, & n-2 \end{vmatrix} = -28(2n-1),$$

again,

$$\begin{vmatrix} n+3, & n-2, & n+1 \\ n-4, & n, & n+4 \\ n-1, & n+2, & n-3 \end{vmatrix} = -80n; \quad \text{therefore } n = -80.$$

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13143. (Professor MATZ.)—A., B., and C. can walk at the rate of 3, 4, and 5 miles an hour. They start from Washington at $m = 1$, $n = 2$, $p = 3$ o'clock p.m., respectively. When B. overtakes A., he is ordered by A. back to C. When will B. and C. meet? Suppose B. had ordered A. back to C., when would A. and C. meet? In case all three continue walking ahead, at what time will they meet?

Solution by Rev. T. WIGGINS, B.A.; Professor SARKAR; *and others.*

Let B. meet A. after x hours; therefore $4x = 3(x+1)$ or $x = 3$; therefore B. meets A. at 5 o'clock; similarly, C. meets A. at 6 o'clock, and C. meets B. at 7 o'clock, on the supposition they all walk ahead.

B. meets A. 12 miles from Washington, and C. is then 10 miles. The distance between B. and C. is 2 miles. Suppose they meet when C. has travelled y miles more; therefore $\frac{1}{3}y = \frac{1}{4}(2-y)$, or $y = 1\frac{1}{7}$; therefore C. has travelled $11\frac{1}{7}$ miles before he meets B., and he does this in $2\frac{2}{7}$ hours; therefore the time is 13 minutes 20 seconds past 5.

If A. were ordered back by B., in the same way it may be shown that C. has walked $11\frac{1}{4}$ miles before he meets A.; therefore the time is 15 minutes past 5.

12995. (G. W. L. WEST.)—Give solutions of $3n^2 - 3n + 1 = \square$.

Solution by H. W. CURJEL, M A.; Colonel CUNNINGHAM; *and others.*

Let $3n^2 - 3n + 1 = x^2$; $\therefore\ (2x)^2 - 3(2n-1)^2 = 1$;

$$\therefore\ x = \frac{(2+\sqrt{3})^m + (2-\sqrt{3})^m}{4} \quad \text{and} \quad 2n-1 = \frac{(2+\sqrt{3})^m - (2-\sqrt{3})^m}{2\sqrt{3}}.$$

[The PROPOSER states that this problem depends on making $3y^2 + 1 = \square$, which can be done by continued fractions.]

11558. (Professor CATALAN.)—K étant un nombre entier, le coefficient de x^K dans le développement de $(1+x)^{2K+2}(1-x)/(1+x^3)$ est 3^K.

Solution by H. W. CURJEL, M.A.; Professor SANJÁNA; *and others.*

Let $(1+x)^{2K+2}(1-x)/(1+x^3) = 1 + \ldots + P_K x^{K-1} + Q_K x^K + R_K x^{K+1} + \ldots$

then ..... (1);

$$(1+x)^{2K}(1-x^2) = (1-x+x^2)(1+\ldots+P_K x^{K-1} + Q_K x^K + R_K x^{K+1} + \ldots)$$

Therefore, equating coefficients of x^{K+1}, we get

$$0 = P_K - Q_K + R_K; \quad \therefore\ P_K + R_K = Q_K.$$

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Again, multiplying both sides of (1) by $1+2x+x^2$, we get

$$1+\ldots+Q_{K+1}x^{K+1}+\ldots = (1+x)^{2(K+1)+2}(1-x)/(1+x^3)$$
$$= 1+\ldots+(P_K+2Q_K+R_K)x^{K+1}+\ldots$$
$$= 1+\ldots+3Q_Kx^{K+1}+\ldots.$$

$\therefore\ Q_{K+1}=3Q_K$; but $Q_1=3$ and $Q_2=3^2$; $\therefore\ Q_K=3^K$.

12988. (J. J. Barniville, B.A.)—Sum the series

$$\frac{1}{1.2}+\frac{1}{6.7}+\frac{1}{11.12}+\ldots.$$

Solution by H. W. Curjel, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Let sum of series $= S$ and $y=F(x)=\frac{1}{1}x-\frac{1}{2}x^2+\frac{1}{3}x^6-\frac{1}{7}x^7+\ldots.$

Then $F(1)=S$, and

$$\frac{dy}{dx}=\frac{1-x}{1-x^5}=\frac{1}{1+x+x^2+x^3+x^4}$$
$$=\frac{1}{2\sqrt{5}}\left\{\frac{2x+\frac{1}{2}(\sqrt{5}+1)}{x^2+\frac{1}{2}(\sqrt{5}+1)x+1}-\frac{2x-\frac{1}{2}(\sqrt{5}-1)}{x^2-\frac{1}{2}(\sqrt{5}-1)x+1}\right.$$
$$\left.+\frac{\frac{1}{2}(\sqrt{5}+1)}{x^2+\frac{1}{2}(\sqrt{5}+1)x+1}+\frac{\frac{1}{2}(\sqrt{5}-1)}{x^2-\frac{1}{2}(\sqrt{5}-1)x+1}\right\}.$$

$$\therefore\ 2\sqrt{5}\,S=\int_0^1\left\{\log\frac{x^2+\frac{1}{2}(\sqrt{5}+1)x+1}{x^2-\frac{1}{2}(\sqrt{5}-1)x+1}+\frac{2(\sqrt{5}+1)}{(10-2\sqrt{5})^{\frac{1}{2}}}\tan^{-1}\frac{x+\frac{1}{4}(\sqrt{5}+1)}{\frac{1}{4}(10-2\sqrt{5})^{\frac{1}{2}}}\right.$$
$$\left.+\frac{2(\sqrt{5}-1)}{(10+2\sqrt{5})^{\frac{1}{2}}}\tan^{-1}\frac{x-\frac{1}{4}(\sqrt{5}-1)}{\frac{1}{4}(10+2\sqrt{5})^{\frac{1}{2}}}\right\}$$
$$=2\log\tfrac{1}{2}(\sqrt{5}+1)+\frac{\sqrt{2}(\sqrt{5}+1)}{(5-\sqrt{5})^{\frac{1}{2}}}\tfrac{1}{10}\pi+\frac{\sqrt{2}(\sqrt{5}-1)}{(5+\sqrt{5})^{\frac{1}{2}}}\tfrac{1}{10}3\pi$$
$$=2\log\tfrac{1}{2}(\sqrt{5}+1)+\tfrac{1}{5}\pi(10-2\sqrt{5})^{\frac{1}{2}}.$$

$$\therefore\ S=1/\sqrt{5}\log\tfrac{1}{2}(\sqrt{5}+1)+\tfrac{1}{50}\pi(50-10\sqrt{5})^{\frac{1}{2}}.$$

8672. (J. O'Regan.)—A. sets out from P to Q at the rate of p miles an hour, and B. at the same time from Q to P at a different rate; they travel uniformly and meet at R. When A. arrives at Q he is detained a hours, and then returns at the rate of $p+q$ miles an hour, and arrives again at R just when B. arrives at P, which is b hours after they met. Find the time in which A. performed the journey from P to Q.

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Solution by H. J. WOODALL, A.R.C.S.; Professor SANJÁNA; *and others.*

Let the distances from P to R, R to Q be x, y miles, respectively.
Let B.'s speed be k miles per hour;

$$\therefore \quad x/p = y/k;$$

and $$b = y/p + a + y/(p+q) = x/k; \quad \therefore \quad x = py/k;$$

$$b = x/k = py/k^2 = y(2p+q)/p(p+q) + a;$$

$$\therefore \quad y = (b-a)p(p+q)/(2p+q); \quad k^2 = (b-a)p^2(p+q)/b(2p+q);$$

$$\therefore \quad (x+y)/p = \left\{(b-a)b(p+q)/(2p+q)\right\}^{\frac{1}{2}} + (b-a)(p+q)/(2p+q).$$

12960. (W. C. STANHAM.)—OA is a horizontal line on an inclined plane; OB is drawn perpendicular to OA, along the plane. At O there is a fine smooth hole through which passes a string connecting two particles P (mass m) and Q (mass $m \sin\alpha$), α being the inclination of the plane. P is held at A and then let go. The first time its path meets OB it cuts it at an angle β, and at the point B. Show that, if the curved line AB revolve round OB, the volume generated

$$= \tfrac{2}{3}\pi \mathrm{OA} \cdot \mathrm{OB}^2/1 + (1+\sin\alpha)\cot^2\beta.$$

Solution by the PROPOSER, Professor KRISHNAMACHARRY, *and others.*

Denote OA and OB by a and b. Let angle between tangent and radius vector be ψ. Then, with usual notation, the equation of work and energy gives

$$\tfrac{1}{2}V^2(1+\sin\alpha\cos^2\psi) = g\sin\alpha(y+a+r) \quad \ldots\ldots\ldots\ldots \quad (1);$$

and for the transversal acceleration

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = g\sin\alpha\cos\theta\, d\theta \quad \ldots\ldots\ldots\ldots \quad (2),$$

or $$V^2p^2 = \int 2g\sin\alpha\, r^3\cos\theta\, d\theta,$$

or $$V^2r^2\sin^2\psi = g\sin\alpha\left(3\int x^2\,dy - x^2y\right).$$

At A, $V = 0$ and $y = 0$; at B, $x = 0$, $y = r = b$; and the volume required is $\int_0^b \pi x^2\,dy = \tfrac{2}{3}\pi ab^2/\left\{1+(1+\sin\alpha)\cot^2\beta\right\}$.

13018. (D. BIDDLE.)—Given the sum of two unknown quantities and the sum of their reciprocals, each sum being real and positive, find, numerically, the cube-root of each original quantity, or, failing that, the sum of their cube-roots, when the product of the two given sums is less than 4.

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Solution by Rev. J. L. KITCHIN, M.A.; Prof. IGNACIO BEYENS; *and others.*

Let $x+y=a$, $x^{-1}+y^{-1}=b$, where $ab<4$; then

$$z^3-3x^{\frac{1}{3}}y^{\frac{1}{3}}z-a=0, \quad \text{where} \quad z=x^{\frac{1}{3}}+y^{\frac{1}{3}}.$$

Now $xy=a/b$; $\therefore$ $z^3-3(a/b)^{\frac{1}{3}}z-a=0$;

$$\therefore \quad z=\left\{\tfrac{1}{2}a+(\tfrac{1}{4}a^2-a/b)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{\tfrac{1}{2}a-(\tfrac{1}{4}a^2-a/b)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

$$=(\tfrac{1}{2}a)^{\frac{1}{3}}\left[1+\left\{(ab-4)/ab\right\}^{\frac{1}{2}}\right]^{\frac{1}{3}}+(\tfrac{1}{2}a)^{\frac{1}{3}}\left[1-\left\{(ab-4)/ab\right\}^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

$$=(\tfrac{1}{2}a)^{\frac{1}{3}}\left\{(1+ip)^{\frac{1}{3}}+(1-ip)^{\frac{1}{3}}\right\}, \quad \text{where } p^2=(4-ab)/ab=\tan^2\alpha, \text{ say,}$$

$$=2\,(\tfrac{1}{8}a)^{\frac{1}{3}}\cos\tfrac{1}{3}\alpha, \quad \text{the sum of the cube roots.}$$

Again, $x-y=(a^2-4a/b)^{\frac{1}{2}}=(a/b)^{\frac{1}{2}}(ab-4)^{\frac{1}{2}}=iap$;

$\therefore$ $(x^{\frac{1}{3}}-y^{\frac{1}{3}})\left\{(x^{\frac{1}{3}}-y^{\frac{1}{3}})^2+3x^{\frac{1}{3}}y^{\frac{1}{3}}\right\}-iap=3$, $y^3+3(\tfrac{1}{2}a)^{\frac{1}{3}}y-iap=0$,

and $y=(\tfrac{1}{2}a)^{\frac{1}{3}}\left\{(1+ip)^{\frac{1}{3}}-(1-ip)^{\frac{1}{3}}\right\}=2\,(a/b)^{\frac{1}{2}}i\sin\tfrac{1}{3}\alpha$.

Therefore the separate cube roots cannot be found.

13123. (W. STANHAM.)—A centre of force C, which attracts matter with a force inversely as the square of the distance, moves in a hyperbola. A thin brittle straight rod has its extremities fixed at the foci. Show that, if the rod breaks (owing to the attraction) at D, CD is parallel to an asymptote of the hyperbola.

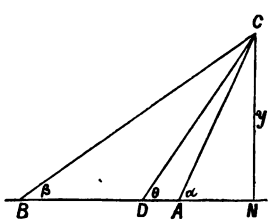
Solution by the PROPOSER, Professor KRISHMACHANDRA DE, *and others.*

Let AB denote the rod; draw CN perpendicular to AB.

Then, if $CN=y$ and $CD=r$,

$$AN=y\cot\alpha, \quad BN=y\cot\beta,$$

$$CD=y\operatorname{cosec}\theta.$$



Let $ND=x$. Then the moment round A, due to the attraction, of an element of the rod at D

$$\propto (x-y\cot\alpha)\sin\theta\,dx/r^2,$$

and the moment round B $\propto (y\cot\beta-x)\sin\theta\,dx/r^2$.

Denoting these by M_1 and M_2, it is clear that, where the tendency of the rod to break is a maximum, $\Sigma_\theta^\alpha M_1=\Sigma_\beta^\theta M_2$ (this follows from the definition of a brittle rod).

Substituting for r, x, and dx in terms of y and θ, we obtain

$$\int_\theta^\alpha(\cot\theta-\cot\alpha)\sin\theta\,d\theta=\int_\beta^\theta(\cot\beta-\cot\theta)\sin\theta\,d\theta.$$

Whence $$\cos\theta=\sin\alpha-\sin\beta)/\sin(\alpha-\beta),$$

or $$\cos\mathrm{CDA}=(\mathrm{BC}-\mathrm{AC})/\mathrm{AB}=1/e,$$

where e is the eccentricity of the path of C. Hence the theorem.

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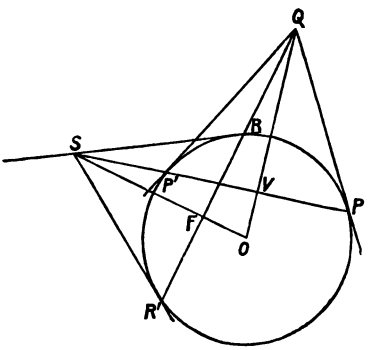
13204. (Rev. T. Wiggins, S.J.)—O is the centre of a circle, Q is a point without it, P, P′ are the points of contact of the tangents from Q, R, R′ are points where the circle is cut by any secant through Q; show that the tangents at R, R′ meet on P, P′ produced.

Solution by W. E. Jeffares, B.A.; C. Bickerdike; *and others.*



From O, the centre of the given circle, draw OF perpendicular to RR′, produce it to meet PP′ in S, and join OQ. OQ is at right angles to PP′. Also

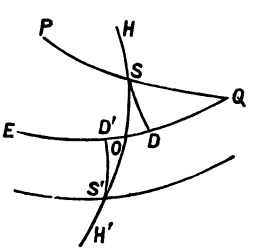
$$QO \times OV = OP^2;$$

but $QO \times OV = OS \times OT$,

since QVFS is a cyclic quadrilateral; therefore S is the pole of RR′. Join SR and SR′. It is easy to see from the equiangular triangles ROS and ROF that RS is a tangent to the circle. Similarly, R′S is a tangent. Therefore the points of concourse of the tangents meet on PP′. [This is a fundamental relation between pole and polar.]

12681. (Professor Wright.)—Find when the Dog Star and the Sun rise together in latitude 42° 30′ North = λ: given the R.A. of Sirius = 6 h. 40 m. 30 sec., and its dec. = 16° 32′ 56″ S.

Solution by Professors Colaw, Nath Coondoo, *and others.*



Let HH′ be the horizon, EQ the equator, PQ the ecliptic, S the sun at rising, S′ Sirius at rising, D′S′ the dec. = δ.

$D'Q = 12H - \text{R.A.} = 79° 52′ 30′ = a'$;

PQE = apparent obliquity of the ecliptic $= 23° 27′ 20″ = B$,

$S'OD' = SOD = 90° - \lambda$,

$SQ = a = 180° - \text{Sun's long.}$

Required DS the Sun's dec. North. Then

$$\sin D'O = \tan\lambda \tan\delta,$$

and $DO = 15° 49′ 2″ = m$ and $OQ = a - m = 64° 3′ 28″ = c$,

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and $$\cot A = \frac{\cos(\beta - \gamma')\cot c}{\cos \gamma'},$$

and $a = 49° 34' 22''$, where $\cot \gamma' = \cot \lambda \cos c$, and $\gamma' = 64° 38' 47''$.

Whence $$\sin DS = \sin B \sin a.$$

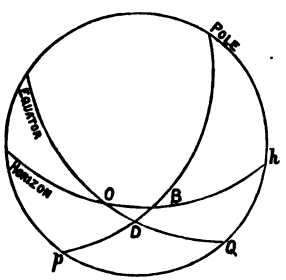
$\therefore$ DS = 17° 38′ 16″ North, and Sun's long. $180° - a = 130° 25' 38''$.

The Sun's dec. and long. found, given by the common solar tables, the date of August 2nd. Eleven days before and after this event is the beginning and end of "Dog Days."

[Mr. Bickerdike gives this solution:—

Let C = colatitude = ∠ BOD,
D = declination = BD,
M = amplitude = OB,
N = ascensional dip = OD,
I = semi-diurnal arc,
R = angle ODB, right angle;

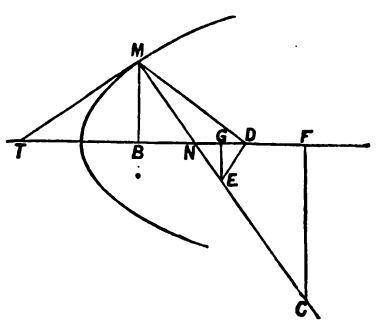
then, by working out from the usual formulæ, he finds the Sun's right ascension to be 1° 44′.



13202. (Professor Izitzéica.)—On considère un point M sur une parabole P. La normale en M, à P, rencontre l'axe de la parabole en N. La perpendiculaire en N, à l'axe, rencontre P en Q et R. Trouver le lieu du centre de gravité du triangle formé par les points de contact des normales à P en M, Q, R avec la développée de la parabole.

Solution by H. W. Curjel, M.A.; Professor Gopalachanar; *and others.*

Let A be the vertex of P. Let tangent at M cut the axis in T. Draw MB perpendicular to the axis and cut off BD from axis = BT. Draw DE at right angles to MD, cutting MN in E. Produce MN to C, making MC = 2ME. Draw CF, EG perpendicular to the axis. Since the tangent at M and the common chord of the parabola and circle of curvature are equally inclined to the axis, it is clear that C is the centre of curvature at M.



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Again $\angle NEG + GED = \angle NED = 2$ rt. $\angle$s $- DMT = 2MTB$
$= 2BMN = 2NEG$;

$\therefore\ NG = GD.$

But $GB = GF$; $\therefore\ NF = BD = TB = 2AB$;

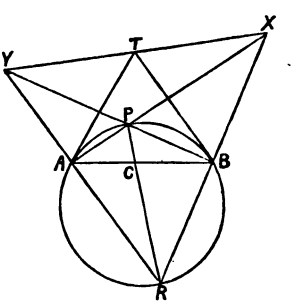
$\therefore\ AN = 3AB + BN = 3AB + 2a.$

Hence the abscissa of the centre of gravity of the $\Delta = 3AB + 6a$ and the ordinate is clearly $= \frac{1}{3}CF$. Hence the locus is a semicubical parabola, which may be obtained from the evolute of P by decreasing the ordinates in the ratio 1 : 3 and moving the curve along the axis through a distance $4a$.

13177. (M. Brierley.)—From a given point draw two tangents to a given circle, and join the points of contact. From the centre of the circle draw any radius (produced, if necessary), perpendicular to which draw a line from the given point, and produce it indefinitely. If, then, lines be drawn from the points of contact of the tangents to intersect in the extremity of the radius before drawn, they will intercept on the indefinite line a portion of it of a constant magnitude.

Solution by J. E. St. Clair; H. W. Curjel, M.A.; *and others.*

Let A, B be the points of contact of tangents from T, P any point on the circle, and XY a line through T perpendicular to the diameter RP. Let RP cut AB in C.



The polar of C passes through T and is perpendicular to RP; $\therefore$ XY is the polar of C. Hence AP, RB meet in X, and BP, RA in Y. Then, since XAY, XBY are right angles, the points A, B lie on the circle of which the diameter is XY.

Now, the pencil (C.XPYA) is harmonic; therefore the poles Y, ∞, X, T form an harmonic range. Hence T bisects XY, and $XY = 2AT =$ constant.

13195. (Professor Honey.)—Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter th forms with the major axis; find the length of this diameter.

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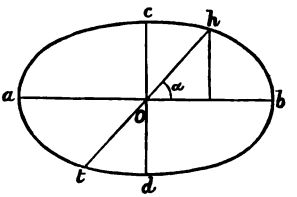
Solution by W. E. JEFFARES, B.A.; Rev. T. WIGGINS, S.J.; *and others.*

The polar equation of an ellipse referred to the centre is

$$r^2 = \frac{a^2b^2}{b^2\cos^2\alpha + a^2\sin^2\alpha};$$

whence the diameter or $2r$

$$= 2ab\,(b^2\cos^2\alpha + a^2\sin^2\alpha)^{-\frac{1}{2}}.$$



13139. (Professor WATSON.)—Find the average area of all right-angled triangles having a given hypotenuse.

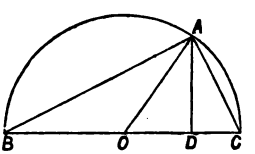
Solution by H. W. CURJEL, M.A.; Prof. MUKHOPADHYAY; *and others.*

Let O be the middle point of the given hypotenuse BC ($= h$). Let $\angle AOC = \theta$.

Then area $= \frac{1}{4}h^2\sin\theta$;

$$\therefore \text{ average area } = \tfrac{1}{4}h^2\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta \Big/ \int_0^{\frac{1}{2}\pi} d\theta$$

$$= h^2/2\pi.$$



8262. (By Professor HANUMANTA RAU, B.A.)—Find the radius of the circle passing through the points whose trilinear coordinates are $(-a, b, c)$, $(a, -b, c)$, and $(a, b, -c)$.

Solution by Professors ZERR, KRISHMACHANDRA DE, *and others.*

P, Q, R, the angular points of the triangle formed by drawing tangents to the circumcircle of ABC at the points A, B, C, are the points required if we regard ABC as the triangle of reference.

$\angle R = \pi - 2C$, $\angle Q = \pi - 2B$, $\angle P = \pi - 2A$,

$RB + BP = c/2\cos C + a/2\cos A = RP$;

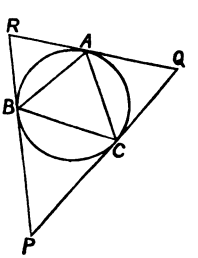
$\therefore$ R = radius $= (c\cos A + a\cos C)$

$/(4\cos A\cos C\sin 2B)$

$\therefore$ R $= b/(4\cos A\cos C\sin 2B)$

$= a/(4\cos B\cos C\sin 2A)$

$= c/(4\cos A\cos B\sin 2C)$.



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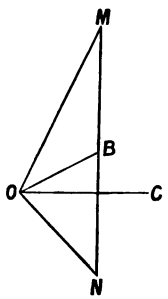
13199. (Professor LAISANT.)—Désignons par (OP, OQ) l'ellipse construite sur les demi-diamètres conjugués OP, OQ. Soient A′, B′, C′ trois foyers des ellipses (OB, OC), (OC, OA), (OA, OB) respectivement. Démontrer que les trois ellipses (OA, OA′), (OB, OB′), (OC, OC′) sont homofocales.

Solution by H. W. CURJEL, M.A.; Prof. KRISHNAMACHARRY; *and others.*

Draw MBN perpendicular to OC, making

$$BM = BN = OC.$$



Then OM = sum and ON = difference of the semi-axes of ellipse (OB, OC), and the major axis is the internal bisector of ∠ MON.

Hence, if OA, OB, OC are represented by the complex quantities α, β, γ, OA′ is represented by

$$\{(\beta + i\gamma)(\beta - i\gamma)\}^{\frac{1}{2}} = (\beta^2 + \gamma^2)^{\frac{1}{2}}.$$

Hence focus of (OA, OA′) is given by $(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}}$.

The symmetry of the result shows that (OA, OA′), (OB, OB′), (OC, OC′) are confocal.

We get a similar result for any number n of straight lines OA, OB, OC, &c., the foci of the last ellipses being $(\Sigma a^2)^{\frac{1}{2}}$, and therefore independent of the order in which the lines OA, OB, &c., are taken.

[OA′² = the difference of the squares of the semi-axes; $\beta^2 + \gamma^2$ = their sum.]

13186. (H. ORFEUR.)—If the denominator of the penultimate convergent of a continued fraction be equal to the numerator of the ultimate convergent, the partial quotients equidistant from the ends are equal.

Solution by the PROPOSER, Professor RAMASWAMI AIYAR, *and others.*

Let the fraction be $\dfrac{1}{b_1+}\ \dfrac{1}{b_2+}\ \dfrac{1}{b_3+}\ \ldots\ldots\ \dfrac{1}{b_n}$.

Let $\{r \ldots s\}$ denote the fraction $\dfrac{1}{b_r+}\ \dfrac{1}{b_{r+1}+}\ \dfrac{1}{b_{r+2}+}\ \ldots\ldots\ \dfrac{1}{b_s}$.

Let $\{2 \ldots (n-2)\} = d/e$, $\{2 \ldots (n-1)\} = h/k$ ..... (1, 2);

therefore
$$\{2 \ldots n\} = \frac{b_n h + d}{b_n k + e} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (2),
$$\{1 \ldots (n-1)\} = \frac{1}{b_1 + h/k} = \frac{k}{b_1 k + h} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (3),
$$\{1 \ldots n\} = \frac{1}{b_1 + \dfrac{b_n h + d}{b_n k + e}} = \frac{b_n k + e}{b_1 (b_n k + e) + b_n h + d} \quad \ldots\ldots\ldots\ldots (5);$$

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therefore, by hypothesis, $b_n k + e = b_1 k + h$.

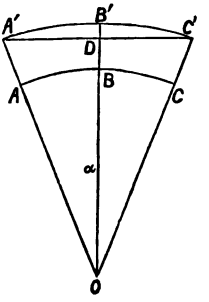
But b_1, b_n, k, e, and h are positive integers, and $k > e$ and also $> h$; therefore $b_n = b_1$ and $e = h$.

But, because $e = h$, the fraction $\{2 \ldots (n-1)\}$ must be similar to $\{1 \ldots n\}$; therefore $b_{n-1} = b_2$, and so on; hence the proposition.

13026. (Rev. T. Roach, M.A.)—On the Bedford Level three poles of equal height are set up in a straight line at distances of 660 yards, and a man looks from the top of the first pole at the top of the third. Show that, owing to the rotundity of the earth, the top of the second pole will be above his line of sight, and calculate the difference approximately, taking the radius of the earth as 3960 miles.

Solution by W. J. Dobbs, M.A.; Rev. J. L. Kitchin; *and others.*

The poles are normal to the surface; hence the tops lie on a circle whose centre is that of the earth. The chord A′C′, the man's line of sight, thus falls below B′.



The angle $\alpha = \dfrac{660}{3960 \times 1760} = \dfrac{1}{10560}$.

Let a be the length of the poles; then

$$OD = (a + 3960 \times 1760) \cos \alpha;$$

$$\therefore \quad DB = (a + 3960 \times 1760)(1 - \cos \alpha)$$

$$= \tfrac{1}{2}\alpha^2 (a + 3960 \times 1760), \text{ approximately,}$$

and, since a is very small compared with the other term in the bracket,

$$= 1/(10560)^2 \times 3960 \times 880$$

$$= \tfrac{1}{3\frac{1}{3}} \text{ yard} = 1\tfrac{1}{8} \text{ inches.}$$

13216. (Franz Rogel.)—Prove that

$$\phi(2) \left\lfloor \frac{n}{2} \right\rfloor + \phi(3) \left\lfloor \frac{n}{3} \right\rfloor + \phi(4) \left\lfloor \frac{n}{4} \right\rfloor \ldots + \phi(n-1) \left\lfloor \frac{n}{n-1} \right\rfloor + \phi(n) \left\lfloor \frac{n}{n} \right\rfloor = \binom{n}{2},$$

wherein $\phi(m)$ denotes the quantity of numbers $< m$ prime to m, and $\left\lfloor \dfrac{n}{r} \right\rfloor$ the greatest integer $\geq \dfrac{n}{r}$.

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Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

The formula is easily verified for small values 2, 3, 4 of n. Call the left-hand side $\psi(n)$. Suppose the formula is true for n, so that

$$\psi(n) = {}^{n}C_2.$$

$$\psi(n+1) = \psi(n) + \Sigma\phi(d) \quad \text{[where the } d\text{'s are the divisors of } n+1$$
$$= {}^{n}C_2 + n = {}^{n+1}C_2. \qquad \text{excluding unity].}$$

Hence, by induction, the formula is universally true.

[The PROPOSER sends the following solution :—

Generally, supposing $A_\nu = +\nu^{r-1}$, there results

$$\sum_{\nu=2}^{m} \nu^{r-1}\phi(\nu)\, B_r\left(1 + \left|\frac{m}{\nu}\right|\right) = \frac{r}{r+1} B_{r+1}(m+1) - B_r(m+1),$$

or, with respect to $\left|\frac{m}{\nu}\right| = 1, \quad \nu > \frac{1}{2}m,$

$$\sum_{\nu=2}^{\nu \leq \frac{1}{2}m} \nu^{r-1}\phi(\nu)\, B_r\left(1 + \left|\frac{m}{\nu}\right|\right) + r^r \sum_{\nu > \frac{1}{2}m}^{m} \nu^{r-1}\phi(\nu)$$
$$= \frac{r}{r+1} B_{r+1}(m+1) - B_r(m+1),$$

wherein $B_r(m)$ denotes the function of BERNOULLI of the rth order.

Finally, putting $A_\nu = (-1)^{\frac{1}{2}(\nu-1)}\nu^r,$

ν an odd number, and introducing the EULER's function of the first order ("Theorie der Euler'schen Functionen," von FRANZ ROGEL, *Sitzungsberichte der Königl. Böhmischen Gesellsch. d. Wissensch.*, Prag, 1893, 1896)

$$E_r(x) = x^r - \binom{r}{2} E_2 x^{r-2} + \binom{r}{4} E_4 x^{r-4} \ldots + \begin{cases} (-1)^{\frac{1}{2}r}\binom{r}{r} E_r \\ (-1)^{\frac{1}{2}(r-1)}\binom{r}{r-1} E_{r-1} x_1 \end{cases}$$

E_{2r} the rth secant coefficient, we get with respect to

$$1^r - 3^r + 5^r \ldots + (-1)^{\frac{1}{2}(k-1)} k^r = \tfrac{1}{2}\left[E_r(0) + (-1)^{\frac{1}{2}(k-1)} E_r(k+1)\right],$$

and $\qquad E_r(0) = (-1)^{\frac{1}{2}r} E_r, \ r \text{ even}; \quad 0, \ r \text{ odd},$

the relation

$$\sum_{\nu=1}^{k} (-1)^{\frac{1}{2}(\nu-1)} \nu^r \phi(\nu) \left\{ E_r(0) + (-1)^{\frac{1}{2}(\lfloor k/\nu \rfloor - 1)} E_r\left(1 + \left|\frac{k}{\nu}\right|\right)\right\}$$
$$= E_{r+1}(0) + (-1)^{\frac{1}{2}(k-1)} E_{r+1}(k+1),$$

or, by respecting $\qquad \left|\frac{k}{\nu}\right| = 1, \quad \nu > \frac{1}{2}k,$

and $\qquad E_r(2) = 2 - (-1)^{\frac{1}{2}r} E_r, \ r \text{ even}; \quad 2, \ r \text{ odd};$

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and separating the cases

(a) r even,

$$\sum_{\nu=2}^{\nu>\frac{1}{2}k} (-1)^{\frac{1}{2}(r-1)} \nu^r \phi(\nu) \left\{ (-1)^{\frac{1}{2}r} \mathrm{E}_r + (-1)^{\frac{1}{2}(\lfloor k/\nu \rfloor - 1)} \mathrm{E}_r \left(1 + \left\lfloor \frac{k}{\nu} \right\rfloor\right) \right\}$$

$$+ 2 \sum_{\nu>\frac{1}{2}k}^{k} (-1)^{\frac{1}{2}(\nu-1)} \nu \ \phi(\nu) = (-1)^{\frac{1}{2}(k-1)} \left\{ \mathrm{E}_{r+1}(1+k) - \mathrm{E}_r(1+k) \right\}$$

$$- (-1)^{\frac{1}{2}r} \mathrm{E}_r,$$

(b) r odd,

$$\sum_{r=2}^{\nu>\frac{1}{2}k} (-1)^{\frac{1}{2}(r+\lfloor k/\nu \rfloor - 2)} \nu^r \phi(\nu) \, \mathrm{E}_r \left(1 + \left\lfloor \frac{k}{\nu} \right\rfloor\right) + 2 \sum_{\nu>\frac{1}{2}k}^{k} (-1)^{\frac{1}{2}(\nu-1)} \nu^r \phi(\nu)$$

$$= (-1)^{\frac{1}{2}(r+1)} \mathrm{E}_{r+1} + (-1)^{\frac{1}{2}(k-1)} \left\{ \mathrm{E}_{r+1}(1+k) - \mathrm{E}_r(1+k) \right\}.]$$

13208. (M. Brierley.)—Given a line drawn from the vertex to the middle of the base, and one of the angles at the base, to construct the triangle when the area is a maximum.

Solution by Rev. T. Wiggins, S.J.; W. E. Jeffares, B.A.; *and others.*

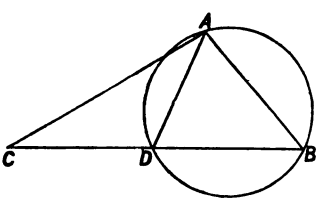
On AD, the given median, describe a segment ABD of a circle containing the given angle.

Bisect the arc ABD in B.

Produce BD to C, making

$$DC = BD.$$

Then ABC is clearly the required triangle.



13171. (D. Biddle.)—Prove that

$$\int_0^a x^n (a-x)^r \, dx = \frac{n!\, r!\, a^{n+r+1}}{(n+r+1)!},$$

a being any quantity, and n, r integers of any magnitude.

Solution by R. Chartres; H. W. Curjel, M.A.; *and others.*

Let a be divided at random into two parts x and y; then

$$a^{n+r} = (x+y)^{n+r},$$

which has $(n+r+1)$ terms; therefore the mean value of each term

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$= a^{n+r} \div (n+r+1)$. But the coefficient of $x^n y^r$ is $\frac{(n+r)!}{r!\,n!}$; therefore the mean value of $x^n y^r = \frac{a^{n+r}\,n!\,r!}{(n+r+1)!}$.

But x has gone through all values from 0 to a; therefore

$$\int_0^a x^n (a-x)^r \, dx = \frac{a^{n+r+1}\,n!\,r!}{(n+r+1)!}.$$

[Mr. CULLEN gives the solution thus:—

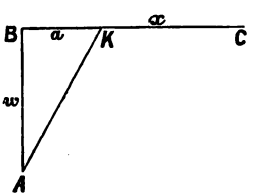
Let $x = az$; then the integral becomes

$$a^{n+r+1} \int_0^1 z^n (1-z)^r \, dz = a^{n+r+1} \mathrm{B}(n+1, r+1)$$

$$= a^{n+r+1} \Gamma(n+1)\,\Gamma(r+1)/\Gamma(n+r+2) = a^{n+r+1} \frac{n!\,r!}{(n+r+1)!}.]$$

13068. (Rev. T. WIGGINS, S.J.)—A person in a boat w miles from the nearest point of the beach wishes to reach in the shortest time a place x miles from that point along the shore. Supposing he can walk y miles an hour, but pull only at the rate of z miles an hour, where must he land? Explain the absence of x in the answer, and the obvious discrepancy when x is less than the distance between the nearest point and the landing point.

Solution by Rev. T. ROACH, M.A.; Professor LAMPE; *and others.*



Let K, the point to which he rows, be distant a from B.

Then time $= \frac{(w^2+a^2)^{\frac{1}{2}}}{z} + \frac{x-a}{y}$,

as this is to be a minimum, we have

$$a = \frac{wz}{(y^2-z^2)^{\frac{1}{2}}}.$$

x does not occur in the answer because the problem is independent of the length of BC, unless $x < \frac{wz}{(y^2-z^2)^{\frac{1}{2}}}$; and then AC is the shortest line.

13172. (R. F. DAVIS, M.A.)—Given S a fixed focus, and OX, OY, fixed directrices; find the envelope of PQ where P, Q are variable points taken one on each directrix equidistant from S. [The PROPOSER desires a verification of his result. Draw SM, SN perpendicular to the directrices; then the envelope is the first negative pedal with respect

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to S of the rectangular hyperbola passing through ONSM. The particular case in which S lies on the bisector of YOX should be noticed.]

Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

Take equations to OX, OY as $y = \pm mx$, and coordinates of S as α, β. Then P and Q are determined by $(x-\alpha)^2+(y-\beta)^2 = r^2$ and $y = \pm mx$.

Taking the positive sign, we get

$$(1+m^2)\,x^2 - 2(\alpha+m\beta)\,x = r^2-\alpha^2-\beta^2;$$

$$\therefore \text{ at P} \qquad \frac{y}{m} = x = \frac{\alpha+m\beta}{1+m^2} + \frac{\left[(m^2+1)\,r^2-(\alpha m-\beta)^2\right]^{\frac{1}{2}}}{1+m^2}.$$

Similarly, at Q, $\dfrac{y}{-m} = x = \dfrac{\alpha-m\beta}{1+m^2} + \dfrac{\left[(m^2+1)\,r^2-(\alpha m+\beta)^2\right]^{\frac{1}{2}}}{1+m^2}$;

each of the square roots having either sign.

Therefore, at R, the foot of perpendicular from S on PQ,

$$x = \frac{\alpha}{1+m^2} + \tfrac{1}{2}\left(\frac{\left[(m^2+1)\,r^2-(\alpha m-\beta)^2\right]^{\frac{1}{2}} + \left[(m^2+1)\,r^2-(\alpha m+\beta)^2\right]^{\frac{1}{2}}}{m^2+1^2}\right),$$

$$y = \frac{m^2\beta}{1+m^2} + \frac{m}{2(m^2+1)}\left\{\left[(m^2+1)\,r^2-(\alpha m-\beta)^2\right]^{\frac{1}{2}} - \left[(m^2+1)\,r^2-(\alpha m+\beta)^2\right]^{\frac{1}{2}}\right\};$$

therefore locus of R is

$$\left(x-\frac{\alpha}{1+m^2}\right)\left(y-\frac{m^2\beta}{1+m^2}\right) = \frac{m}{4(m^2+1)^2}\,(4\alpha\beta m),$$

i.e.,
$$(m^2+1)\,xy = y\alpha + m^2x\beta,$$

a rectangular hyperbola through S, M, O, N.

Locus of R is also clearly the pedal with respect to S of the envelope of PQ. When $\beta = 0$, locus of R consists of

$$y = 0 \quad \text{and} \quad x = \alpha/(m^2+1),$$

and the envelope of PQ degenerates into the point at infinity on the axis of y and a parabola focus S and tangent at vertex MN.

[*Otherwise*:—Let x, y be the coordinates of R referred to OX, OY. Then $OP = 2x$, $OQ = 2y$; also

$$OM^2-PM^2 = SO^2-SP^2 = SO^2-SQ^2 = ON^2-QN^2.$$

Thus $x^2-y^2 = OM\,.\,x-ON\,.\,y$.]

3827. (Professor MUKHOPADHYAY.)—One end of a rope is tied to a stake driven into the side of a square 5 acre field, 12 yards from one corner, and the other is fastened to the hind leg of a cow, whose length is 6 feet, so as to allow the animal to graze over one-eighth part of the field. Required the length of the rope.

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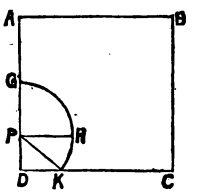
Solution by Professors ZERR, KRISHMACHANDRA DE, *and others.*



Let P be the point where the rope is fastened, $\angle HPK = \theta$, PH parallel to DC, x = length of rope.

$\therefore$ PG = PH = PK = $x + 6$ feet = $(x + 2)$ yds.

$\therefore$ $x + 2 = 12/\sin\theta$, DK = $12\cot\theta$

Area = area PHG + area KPH + area PDK.

$\therefore$ $36\pi/\sin^2\theta + 72\theta/\sin^2\theta + 72\cot\theta = 3025$.

$\therefore$ $36\pi + 72\theta + 36\sin 2\theta = 3025\sin^2\theta$

$\therefore$ $\theta = 12° 36' 55''$ nearly.

$x + 2 = 12/\sin\theta = 54{\cdot}942$ yds.

$\therefore$ $x = 52{\cdot}942$.

12760. (JOHN GRIFFITHS, M.A.) — Show that in a right-angled triangle the nine-point circle is a G.B. circle (see *Proceedings of the London Mathematical Society*, June, 1894).

Solution by the PROPOSER, Professor CHAKRAVARTI, *and others.*

The equation of a G.B. circle may be written in the form

$$x\sin B\sin C\left\{\operatorname{cosec}^2 A - (\cot\omega - \cot\Omega)^2\right\} + y\sin B(\cot\omega - \cot C) + z\sin C(\cot\omega - \cot B) = 2\operatorname{cosec} A\sin B\sin C\cot\omega,$$

where Ω is the BROCARD angle of triangle of reference ABC. (See *Proceedings of the London Mathematical Society*, Vol. XXVI., Nos. 510, 511.)

If $B = \frac{1}{2}\pi$ and $\omega = C$,

this reduces to $x\cos A + z\cos C = 2$,

which represents the nine-point circle of the triangle.

13174. (W. C. STANHAM.)—Show that $\int_0^\infty y\,dx$ has the same value for the two curves $xy^2 = f(x + 2a)$ and $xy^2 = f(x + a^2/x)$.

Solution by Rev. J. CULLEN, B.A.; *the* PROPOSER; *and others.*

To show that $\int_0^\infty f_1(x + 2a)\,(dx/\sqrt{x}) = \int_0^\infty f_1(x + a^2/x)\,(dx/\sqrt{x})$,

take a term of the form $(x + 2a)^n$ in $f_1(x + 2a)$; then there will be a term

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$(x+a^2/x)^n$ in $f_1(x+a^2/x)$. Let

$$I = \int_0^\infty (x+2a)^n (dx/\sqrt{x}),\ J = \int_0^\infty (x+a^2/x)^n (dx/\sqrt{x}) \ \ldots\ (1), (2).$$

Putting $x = 2a\tan^2\theta$ in (1), $x = a\tan\theta$ in (2), and $\frac{1}{2}(2n+1) = -m$, we have

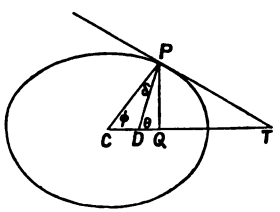
$$I = a^{-m}.\,2^{1-m}\,\Gamma(m)\,\Gamma(\tfrac{1}{2})/2\Gamma(m+\tfrac{1}{2}),$$
$$J = a^{-m}\,\Gamma(\tfrac{1}{2}.m)\,\Gamma\tfrac{1}{2}(m+1)/2\Gamma(m+\tfrac{1}{2}).$$

But, by Quest. 9066, Vol. XLVIII., we find that I = J. Therefore, &c.

12542. (Professor SCHEFFER.)—According to BESSEL the ratio of the square of the polar diameter of the earth to that of the equatorial diameter is ·9933254. Find (1) at what altitude the angle made by a body falling to the earth with a perpendicular to the surface is greatest; and (2) also this maximum angle.

Solution by Professors COLAW, MUKHOPADHYAY, *and others.*

Let C be the centre of the ellipse which a vertical section through the earth will form, PT a tangent, PD the corresponding normal; then $\angle PDT = \theta$ is called the geographical latitude, and $\angle PCT = \phi$ the geocentric latitude, and for $\angle CPD = \delta$ we have to find the maximum value.



Putting CQ = x, PQ = y, we have

$$\tan\theta = -(dx/dy) = a^2y/b^2x;$$

but $y/x = \tan\phi$; $\therefore$ $\tan\phi = (b^2/a^2)\tan\theta$;

and, since $\delta = \theta - \phi$, we have

$$\tan\delta = \tan(\theta-\phi) = \left\{(a^2-b^2)\tan\theta\right\} / (a^2+b^2\tan^2\theta).$$

Differentiating, we find, for a maximum value of δ, $\tan\theta = a/b$, and then

$$\tan\delta = \tfrac{1}{2}.(a^2-b^2)/ab = \tfrac{1}{2}(a/b - b/a).$$

Substituting ·9933254 for b^2/a^2, we find

$$\theta = 45^\circ\,5'\,45'' \quad \text{and} \quad \delta = 11'\,30''.$$

3745. (Rev. Dr. BOOTH, F.R.S.)—If α and β are semi-axes of the evolute of an ellipse, show that there is but one length of flexible radius or string that will describe an ellipse, and that the end of any other length but $\frac{\beta^3}{\beta^2-\alpha}$ will describe a curve of a higher order than the second.

Solution by Profs. RAMACHANDRA RAU, KRISHNAMACHARRY, *and others.*

If ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the evolute is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

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$$\therefore \quad a = \frac{a^2-b^2}{a} \quad \text{and} \quad \beta = \frac{a^2-b^2}{b};$$

$$\therefore \quad l \equiv \frac{\beta^3}{\beta^2-a^2} = \frac{a^2}{b} = \text{radius of curvature at extremity of minor axis.}$$

So, if a string of length l be laid along the axis from one of the cusps and wound along the evolute, the extremity traces the ellipse.

If the length be other than l, the required locus is the locus of points chosen on the normals so that the distance from the point whence the normal is drawn is constant. The required locus is a curve of higher degree than the second unless the distance = 0.

13214. (R. F. Muirhead.)—Prove the following formulæ :—

$$\frac{(2n)!}{(2r)!\,(2n-2r)!} = n!\left\{\frac{1}{r!\,(n-r)!} + \frac{2^2}{2!\,(r-1)!\,(n-r-1)!} + \frac{2^4}{4!\,(r-2)!\,(n-r-2)!} + \dots\right\},$$

$$\frac{(2n)!}{(2r+1)!\,(2n-2r-1)!} = n!\left\{\frac{2}{1!\,r!\,(n-r-1)!} + \frac{2^3}{3!\,(r-1)!\,(n-r-2)!} + \frac{2^5}{5!\,(r-2)!\,(n-r-3)!} + \dots\right\},$$

where the series in each case is to be continued to the term which contains 0! (read as unity) as a factor in its denominator.

Solution by H. W. Curjel, M.A.; Prof. Mukhopadhyay; *and others.*

By classifying the combinations $2r$ and $2r+1$ at a time of $2n$ things which are arranged in pairs a_1a_2, b_1b_2, &c., according as they contain r, $(r-1)$, $(r-2)$, ..., 2, 1, 0 pairs, we see that

$$\frac{2n!}{2r!\,(2n-2r)!} = {}^{2n}C_{2r} = {}^nC_r + 2^2\,{}^nC_2\,{}^{n-2}C_{r-2} + 2^4\,{}^nC_4\,{}^{n-4}C_{r-2} + \dots$$

$$= n!\left\{\frac{1}{r!\,(n-r)!} + \frac{2^2}{2!\,(r-1)!\,(n-r-1)!} + \frac{2^4}{4!\,(r-2)!\,(n-r-2)!} + \dots\right\}$$

$$\frac{2n!}{(2r+1)!\,(2n-2r-1)!} = {}^{2n}C_{2r+1}$$

$$= 2.\,{}^nC_1\,{}^{n-1}C_r + 2^3\,{}^nC_3\,{}^{n-3}C_{r-1} + 2^5\,{}^nC_5\,{}^{n-5}C_{r-2} + \&c.$$

$$= n!\left\{\frac{2}{1!\,r!\,(n-r-1)!} + \frac{2^3}{3!\,(r-1)!\,(n-r-2)!} + \frac{2^5}{5!\,(r-2)!\,(n-r-3)!} + \&c.\right\}.$$

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6494. (Professor ARTEMAS MARTIN, LL.D.)—An auger hole, radius b, is made through a sphere, radius r, at random; find the average volume removed.

Solution by H. J. WOODALL, A.R.C.S.; Professor COONDOO; *and others.*

Take axis of y in the plane of, and perpendicular to, the axis of auger hole, and passing through the centre of the sphere. Axis of z, along axis of anger hole. Axis of w perpendicular to both. Let distance of axis of auger hole from centre of sphere be x.

Then volume of hole (*vide* limits of y)

$$= 4\int_{-b}^{+b}\int_{0}^{(b^2-y^2)^{\frac{1}{2}}}\left\{r^2-(x+y)^2-w^2\right\}dw\,.\,dy.$$

Average volume is obtained by multiplying by $x\,dx$ and integrating between limits $r-b$ and 0, and dividing by $\int x\,dx = \frac{1}{2}(r-b)^2$.

13193. (Professor COCHEZ.)—Limite de

$$\left(1+\frac{2}{\sqrt{(m^2+1)}}\right)^{\sqrt[3]{(3m^3+1)}} \quad \text{pour } m=\infty.$$

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; *and others.*

Let limit $= u$. Then we have

$$\log u = \text{limit of } m\sqrt[3]{3}\left(1+\frac{1}{3m^3}\right)^{\frac{1}{3}}\log\left(1+\frac{2}{m(1+1/m^2)^{\frac{1}{2}}}\right) = 2\sqrt[3]{3}.$$

$$\therefore\quad u = e^{2\sqrt[3]{3}}.$$

12547. (Professor NEWTON.)—The axes of two right cylinders whose bases are circles of 4 and 6 inches radius respectively intersect at right angles; compute to four decimal places the lengths of the curves of intersection of the two surfaces.

Solution by Professors COLAW, KRISHNAMACHARRY, *and others.*

Make $m = 6$ inches, $n = 4$ inches; then the Cartesian equations of the cylinders are $z^2+x^2=m^2$, $y^2+x^2=n^2$ .....(1, 2);

$$\therefore\quad \frac{dz}{dx}=-\frac{x}{z}=-\frac{x}{\sqrt{(m^2-x^2)}},\qquad \frac{dy}{dx}=-\frac{x}{y}=-\frac{x}{\sqrt{(n^2-x^2)}} \quad ...(3, 4).$$

Hence the expression for the lengths of the curves of intersection of the

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two surfaces becomes (TODHUNTER's *Integral Calculus*, p. 116)

$$L = 4\int_0^m \sqrt{\left(1+\frac{x^2}{m^2-x^2}+\frac{x^2}{n^2-x^2}\right)}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Make $c^2 = n^2/m^2$, and $x = n\sin\phi$; then $dx = n\cos\phi\,d\phi$. Transforming (5), &c.,

$$L = 4\int_0^m \sqrt{\left(\frac{m^2n^2-x^4}{(m^2-x^2)(n^2-x^2)}\right)}\,dx = 4n\int_0^{\frac{1}{2}\pi} \sqrt{\left(\frac{1-c^2\sin^4\phi}{1-c^2\sin^2\phi}\right)}\,d\phi$$

$$= 4n\int_0^{\frac{1}{2}\pi} \left[1+\tfrac{1}{2}c^2\sin^2\phi+(\tfrac{3}{8}c^4-\tfrac{1}{2}c^2)\sin^4\phi+(\tfrac{5}{16}c^6-\tfrac{1}{4}c^4)\sin^6\phi+\&c.\right]d\phi$$

$$(-\tfrac{1}{8}c^4\sin^8\phi \text{ comes later on, but no more } c^4)$$

$$= 2\pi n\left[1+\frac{c^2}{16}+\frac{29c^4}{1024}+\&c.\right] = 25{\cdot}97+\text{inches}.$$

13178. (Rev. T. WIGGINS, S.J.)—A candidate is examined in four papers, to each of which is assigned m marks as a maximum. His total on the four papers is $\frac{11}{20}m$. In how many ways can this occur? Supposing there were n papers with a maximum of m marks each, and that his total was q: can a solution be found to this.

Solution by H. W. CURJEL, M.A.; Professor SARKAR; *and others.*

Let $m = 5n$. The number of ways is clearly the coefficient of x^{11m} in $(1+x+x^2+\ldots+x^m)^4$, *i.e.*, in

$$(1-4x^{m+1}+6x^{2m+2}-4x^{3m+3}+x^{4m+4})$$

$$\times\left(1+4x+\ldots+\frac{(r+1)(r+2)(r+3)}{3!}x^r+\ldots\right)$$

$$= \frac{(2m+n+1)(2m+n+2)(2m+n+3)-4(m+n)(m+n+1)(m+n+2)}{3!}$$

$$+(n-1)n(n+1).$$

In the second case the number of ways

$$= \text{coefficient of } x^q \text{ in } \left(\frac{1-x^{m+1}}{1-x}\right)^n$$

$$= \frac{(q+1)(q+2)(q+3)\ldots(q+n-1)}{(n-1)!} - \frac{n}{1}\frac{(q-m)(q-m+1)\ldots(q-m+n-2)}{(n-1)!}$$

$$+\frac{n(n-1)}{1.2}\frac{(q-2m-1)(q-2m)\ldots(q-2m+n-3)}{(n-1)!} - \&c.,$$

to $I\left(\frac{q+1}{m+1}+1\right)$ terms.

13038. (Professor GOPALACHANAR.) — Prove that the difference in length of the two loops of the limaçon $r = 2\cos\theta+1$ is 8.

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Solution by Rev. J. CULLEN, B.A.; Prof. MUKHOPADHYAY; *and others.*

Let δ be the required difference; then

$$\tfrac{1}{2}\delta = \left\{\int_0^{\frac{1}{2}\pi} - \int_{\frac{1}{3}\pi}^{\pi}\right\}(9-8\sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}}\,d\theta;$$

$$\therefore\quad \delta = 12\left\{2\mathrm{E}(\tfrac{2}{3}\sqrt{2},\ \tfrac{1}{3}\pi) - \mathrm{E}(\tfrac{2}{3}\sqrt{2},\ \tfrac{1}{2}\pi)\right\}.$$

But the amplitudes and modulus satisfy the conditions for the addition equation of elliptic integrals of the second kind;

$$\therefore\quad \delta = 12\times\tfrac{2}{3}\times\sin^2\tfrac{1}{3}\pi = 8.$$

13168. (Professor NATH COONDOO.)—Prove that the whole arc of the curve $8a^2y^2 = x^2(a^2-2x^2)$ is πa; and for the part included in the positive quadrant the centre of gravity of the area is $\left(\frac{3\pi a}{16\sqrt{2}},\ \frac{1}{10}a\right)$; the centre of gravity of the arc is $\left(\frac{5a}{3\sqrt{2}\pi},\ \frac{a}{4\pi}\right)$; the centre of gravity of the volume generated by revolution about the axis of x is $\left(\frac{5a}{8\sqrt{2}},\ 0\right)$; and that of the area of the surface generated is $\left(\frac{3a}{5\sqrt{2}},\ 0\right)$.

Solution by Rev. J. CULLEN, B.A.; Prof. KRISHNAMACHARRY; *and others.*

Let $x\sqrt{2} = a\sin\theta$; then $4y = a\sin\theta\cos\theta$; therefore

$$dy/dx = \cos 2\theta/2\sqrt{2}\cos\theta.$$

Hence $$s = a/2\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}(\cos 2\theta + 2)\,d\theta = \pi a.$$

For the c.g. of the area $\bar{x}\int y\,dx = \int xy\,dx$;

therefore $$\bar{x}\sqrt{2}\int_0^{\frac{1}{3}\pi}\sin\theta\cos^2\theta\,d\theta = a\int_0^{\frac{1}{3}\pi}\sin^2\theta\cos^2\theta\,d\theta,$$

or $$\bar{x}\sqrt{2}\,\Gamma(3) = a\Gamma(\tfrac{3}{2})\,\Gamma(\tfrac{5}{2});\qquad \therefore\quad \bar{x} = 3\pi a/16\sqrt{2}.$$

In the same manner, $\bar{y}\,.\,8\int_0^{\frac{1}{3}\pi}\sin\theta\cos^2\theta = a\int_0^{\frac{1}{3}\pi}\sin^2\theta\cos^3\theta\,d\theta,$

or $$\bar{y}\,.\,8\Gamma(\tfrac{7}{4}) = a\Gamma(\tfrac{5}{4});\qquad \therefore\quad \bar{y} = a/20.$$

Employing the usual formulæ in this way and taking the limits throughout as $\frac{1}{2}\pi$ and 0, we find for the arc

$$\bar{x}\sqrt{2}\,\pi = a\left[\Gamma(\tfrac{3}{2})/2\,\Gamma(\tfrac{5}{2}) - \Gamma(\tfrac{1}{2})/2\,\Gamma(\tfrac{5}{2}) + 2\right];$$

$$\therefore\quad \bar{x} = 3a/5\sqrt{2}\,\pi,\quad \bar{y}\,.\,4\pi = a\int_0^{\frac{1}{3}\pi}\cos\theta\sin\theta\,(\cos 2\theta + 2)\,d\theta = a.$$

For the volume $\bar{x}\sqrt{2}\,\Gamma(4)\,\Gamma(\tfrac{3}{2}) = a\Gamma(\tfrac{7}{2});\qquad \therefore\quad \bar{x} = 5a/8\sqrt{2},\quad \bar{y} = 0.$

For the surface

$$\bar{x}\sqrt{2} = a\left[\Gamma(\tfrac{3}{2})/2\,\Gamma(\tfrac{7}{2}) - \Gamma(\tfrac{5}{2})/2\,\Gamma(\tfrac{7}{2}) + \Gamma(\tfrac{3}{2})/\Gamma(\tfrac{5}{2})\right] = 3a/5,\quad \bar{y} = 0.$$

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11154. (Professor SYLVESTER.)—Let us understand by the discriminant of a *form* the function whose evanescence is the condition of the equality of the roots in which the coefficient of the term in which only the first and last coefficients of the form appear as unity, and by the discriminant of a rational integral function what the discriminant of its form becomes when the coefficients of that form are the coefficients of the function.

Prove that the discriminant of the function

$$x^n + x^{n-1} + x^{n-2} + \ldots + x + 1,$$

where n is any number greater than unity, is equal to $\frac{(n+1)^{n-1}}{n}$, and show that it is always less than, but continually converges to, a ratio of equality with e/n as n increases indefinitely.

Solution by H. W. CURJEL, M.A.; Professor SANJÁNA; *and others.*

Let $\mathrm{E}(u, v)$ be the eliminant of u and v, obtained by the dialytic method. Then $\mathrm{E}(u_n, du_n/dx)$ is clearly $= n^n \times$ discriminant of u_n, as defined in the question, where u_n is a function of the nth degree in x.

Let $$\mathrm{E}(u_n, du_n/dx) = \mathrm{D}u_n.$$

Now, since $\mathrm{E}(u, v)$ may also be found as $\Pi(\alpha - \beta)$, α being a root of $u = 0$ and β of $v = 0$, and $\mathrm{D}(u)$ is the product of the squared differences of the roots of $u = 0$,

$$\mathrm{D}(uv) = \mathrm{D}(u) \times \mathrm{D}(v) \times \{\mathrm{E}(u, v)\}^2;$$

$$\therefore\ \mathrm{D}(x^{n+1}-1) = \mathrm{D}(x^n + x^{n-1} + \ldots + 1) \times \mathrm{D}(x-1) \times \{\mathrm{E}[x-1, (x^{n+1}-1)/(x-1)]\}^2.$$

Now $(d/dx)(x^{n+1}-1) = (n+1)x^n$; $\therefore\ \mathrm{D}(x^{n+1}-1) = (n+1)^{n+1}$,

and $\mathrm{D}(x-1) = 1$ and $\mathrm{E}[x-1, (x^{n+1}-1)/(x-1)] = n+1$;

$$\therefore\ \mathrm{D}(x^n + x^{n-1} + x^{n-2} + \ldots + 1) = (n+1)^{n-1};$$

$$\therefore\ \text{discriminant} = (n+1)^{n-1}/n^n = [(n+1)/n]^n/(n+1).$$

Now, if $m > n$, $\qquad a^{m/n} - 1 > m/n \,.\, (a-1)$;

let $\qquad a = (m+1)/m$;

$$\therefore\ [(m+1)/m]^{m/n} - 1 > 1/n;\ \therefore\ [(m+1)/m]^m > [(n+1)/n]^n;$$

but, when m is infinite limit of $[(m+1)/m]^m = e$,

$$\therefore\ \text{discriminant} < e/(m+1) < e/n.$$

When n is infinite, discriminant $= e/(n+1)$, the ratio of which to e/n, when n is ∞, is unity.

13030. (R. F. DAVIS, M.A.)—Two circles intersect orthogonally at B, C, and EF is the diameter of the first circle passing through the centre of the second. Through any point U on the second circle draw EU, FU meeting the first again in P, Q. Prove that a third circle can be inscribed in the quadrilateral PBQC, whose centre is at U. Hence

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show that the rectangle contained by the diagonals of any quadrilateral inscribed in one fixed circle, and circumscribed to another fixed circle, is constant. (Compare Quest. 12974.) [The PROPOSER states that there appeared in the *Reprint*, many years ago, a semi-geometrical proof (by himself) of the relation connecting the distance between the centres and the radii of two circles such that quadrilaterals could be inscribed in the one and circumscribed to the other. It was reprinted in CASEY'S *Trigonometry*, and this is an *extension* of the idea of the proof.]

Solution by the PROPOSER, Professor CHAKRAVARTI, *and others.*

Here we have

$$2\theta + 2\phi = \pi,$$

$$\phi = \tfrac{1}{2}\pi - \theta;$$

hence

$$UE : UF = CE : CF$$

$$= \sin\theta : \sin\phi.$$

But

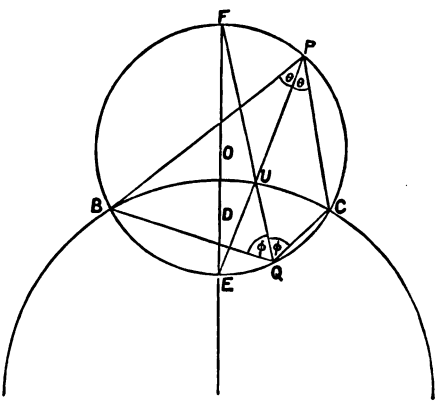
$$UE : UF = UQ : UP;$$

$$UP \sin\theta = UQ \sin\phi;$$

$$\therefore \frac{PQ}{EF} = \frac{UP}{UF} = \frac{UP.UQ}{UF.UQ}$$

$$= \frac{r^2 . \operatorname{cosec}\theta . \operatorname{cosec}\phi}{R^2 - \delta^2}$$

$$= \frac{2r^2}{(R^2 - \delta^2)\sin 2\theta};$$



$$\therefore \frac{PQ}{2R} = \frac{2r^2 . 2R}{(R^2 - \delta^2) . BC}; \quad \therefore \quad PQ . BC = \frac{8R^2 r^2}{R^2 - \delta^2}.$$

13170. (EDITOR.)—If $y = x^2 - 1$, prove that

$$y \frac{d^{n+1} y^n}{dx^{n+1}} = n(n+1) \frac{d^{n-1} y^n}{dx^{n-1}};$$

$$y \frac{d^{m+2} y^n}{dx^{m+2}} + 2(m - n + 1) x \frac{d^{m+1} y^n}{dx^{m+1}} = (m+1)(2n - m) \frac{d^m y^n}{dx^m}.$$

Solution by Professor SANJÁNA; Rev. J. CULLEN, B.A.; *and others.*

We have $(x^2 - 1) \frac{dy^n}{dx} = (x^2 - 1) . n y^{n-1} . 2x = 2nxy^n;$

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differentiating $m+1$ times, employing LEIBNITZ's Theorem, thus:

$$\frac{d^{m+2}y^n}{dx^{m+2}}(x^2-1)+(m+1)\frac{d^{m+1}y^n}{dx^{m+1}}.2x+\frac{(m+1)\,m}{2}\frac{d^my^n}{dx^m}.2$$

$$=2n\left\{\frac{d^{m+1}y^n}{dx^{m+1}}.x+\frac{d^my^n}{dx^m}(m+1)\right\};$$

$$\therefore\quad y\frac{d^{m+2}y^n}{dx^{m+2}}+2(m-n+1)x\frac{d^{m+1}y^n}{dx^{m+1}}=(m+1)(2n-m)\frac{d^my^n}{dx^m}.$$

The first result follows by putting $m+1=n$ in above.

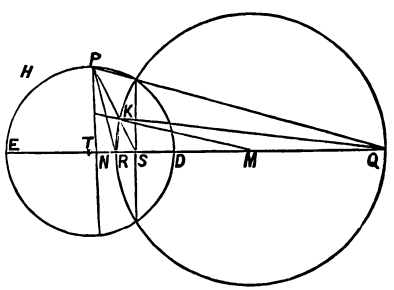
13162. (Professor NEUBERG.)—Construire un triangle ABC, connaissant les longueurs AB, AC de deux côtés et sachant que les symédianes issues de B et C sont rectangulaires.

Solution by H. W. CURJEL, M.A.; Professor MUKHOPADHYAY; *and others.*

Take any base QR and divide it internally at D and externally at E in the ratio AB : AC; bisect QR, DE in M, T, and on QR, DE as diameters describe circles QRG, DEH. Let their common chord cut QR in S. Then the vertex P of $\triangle$PQR, in which

PQ : PR = AB : AC,

lies on circle DEH, and PS will be a symmedian. Let PQR be a triangle such that its symmedian point K lies on $\odot$QGR. Draw PN perpendicular to QR. Then MK bisects PN.



Take M as origin, and ME as axis of x.

Let MR = r, MT = k, MS = d, and MN = x;
PN = y, and KMR = θ.

Then $$\frac{x-d}{r\cos\theta-d}=\frac{y}{r\sin\theta},\qquad\frac{2x}{\cos\theta}=\frac{y}{\sin\theta},$$

and $$(x-k)^2+y^2=k^2-r^2\quad\text{and}\quad kd=r^2;$$

$$\therefore\quad x^2=\frac{2r^4}{3r^2-k^2}=\frac{2r^2d^2}{3d^2-r^2}.$$

This determines P. $3r^2$ must $> k^2$; condition that MN $<$ ME gives $5r>3k$ or QP $>$ 2PR, *i.e.*, AB $>$ 2AC.

Hence $\triangle$ABC may be constructed similar to PQR.

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13175. (R. KNOWLES, B.A.)—PQ is the chord of contact of tangents to a parabola, focus F, drawn from T; show that (1) the centre O of the circle TPQ lies on the circle PQF; (2) OF and TF are at right angles; (3) if PQ passes through a fixed point on the axis, the envelope to the tangent to the circle at O is a parabola.

Solution by H. W. CURJEL, M.A.; Prof. RAMASWAMI AIYAR, M.A.; *and others.*

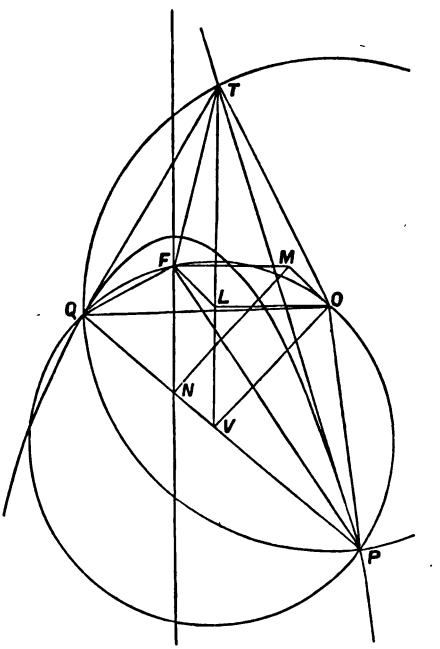
$\angle POQ = 2\angle PTQ$
$= \angle PFQ;$
$\therefore$ P, O, F, Q are concyclic.

Again,
$\angle OFP = \angle OQP$
$= \angle OPQ$
= angle between OF produced and FQ;
$\therefore$ FO and FT bisect $\angle PFQ$; $\therefore$ OFT is a right angle.

Let V be the middle point of PQ, and let PQ cut the axis in N.

Draw FM at right angles to the axis, cutting the tangent at O in M. Draw OL perpendicular to TV.

$\angle LFO = \angle LTO$
$= \angle OPF$
$= \angle FOM;$
$\therefore$ FL is parallel to OM, and therefore to VN;
$\therefore$ OM = FL = VN;
$\therefore$ MN is parallel to OV, and is therefore perpendicular to OM; $\therefore$ envelope of OM is a parabola, focus N and vertex F.



3984. (Professor MINCHIN, M.A.)—If E is the complete elliptic function of the second kind, with modulus k, and if k' is the complementary modulus, prove that

$$E = \frac{\pi k'^2}{2}\left\{1 + \Sigma(2n+1)\left(\frac{.3.5\ldots 2n-1}{2.5.6\ldots 2n}.k^n\right)^2\right\},$$

n assuming all values from 1 to ∞.

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Solution by Rev. J. CULLEN, B.A.; Professor GOPALACHANAR; *and others.*

Expanding $(1-k^2\sin^2\theta)^{\frac{1}{2}}$ in K and integrating each term, we have

$$K = \frac{\pi}{2}\left\{1+(\tfrac{1}{2})^2k^2+\left(\frac{1.3}{1.4}\right)^2k^4+\left(\frac{1.3.5}{2.4.6}\right)^2k^6+\&c.\right\};$$

but $$E_1 = k'^2\left\{K+k\frac{dK}{dk}\right\}$$ (CAYLEY, *E.F.*, Art. 73).

Hence the above expression for E.

13163. (Professor SANJÁNA.)—On the same circle as base and on the same side of it are described a hemisphere and a cone of vertical angle $(2\alpha < 90°)$; find the volume of the hemisphere outside the cone, and of the cone outside the hemisphere.

Solution by Rev. T. WIGGINS, B.A.; Professor GOPALACHANAR; *and others.*

Volume of hemisphere is $\frac{2}{3}\pi r^3$, volume of cone is $\frac{1}{3}(\pi r^3)\cot\alpha$.

Volume of cone CEG is

$$\tfrac{1}{3}(\pi\times EF^2\times CF) = \tfrac{1}{3}(\pi\times EF^3)\cot\alpha$$
$$= \tfrac{1}{3}(\pi\times r^3\cos^3 2\alpha)\cot\alpha,$$

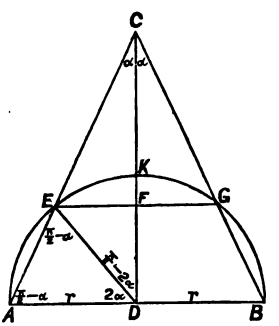
because $EF = r\cos 2\alpha$.

Volume of cap

$$EKGF = \pi\times KF^2(r-\tfrac{1}{3}KF)$$
$$= \pi\times(KD-FD)^2\{r-\tfrac{1}{3}(KD-FD)\}$$
$$= \tfrac{1}{3}(\pi\times r^3)(1-\sin 2\alpha)(2+\sin 2\alpha);$$

$\therefore$ volume of cone outside the sphere is

$$\tfrac{1}{3}\pi r^3\{\cos^3 2\alpha\cot\alpha-(1-\sin 2\alpha)(2+\sin 2\alpha)\}.$$



The volume of the sphere outside the cone is

volume of hemisphere − volume of frustum EAGB − volume of cap EKGF

$$= \tfrac{2}{3}\pi r^3-\tfrac{1}{3}(\pi r^3)\cot\alpha+\tfrac{1}{3}(\pi r^3)\cos^3 2\alpha\cot\alpha-\tfrac{1}{3}(\pi r^3)(1-\sin 2\alpha)(2+\sin 2\alpha).$$

13181. (H. W. SEGAR, B.A.)—Show that

$$\left(a^2\frac{d}{da}+b^2\frac{d}{db}+\ldots\right)H_n = (n+1)H^{n+1}-H_1H_n,$$

where H_n is the sum of the homogeneous products of the quantities $a, b, c, \ldots$ of degree n. Also find the result of n operations.

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Solution by Rev. J. CULLEN, B.A.; Professor MUKHOPADHYAY; *and others.*

Let $$P = \{(1-ax)(1-bx)(1-cx)\ldots\}^{-1}$$
$$= 1 + H_1x + H_2x^2 + \ldots + H_nx^n + \ldots,$$

and $$\nabla P \equiv \left(a^2\frac{d}{da} + b^2\frac{d}{db} + \ldots\right) P.$$

Then we have $$\nabla P = P\Sigma\{a^2x/(1-ax)\};$$

also $$\frac{d}{dx}\log P = \frac{1}{P}\frac{dP}{dx} = \Sigma\{a/(1-ax)\} = \Sigma\{a + a^2x/(1-ax)\};$$

$$\therefore \quad \nabla P = \frac{dP}{dx} - H_1P.$$

Equating the coefficients of x^n, we obtain the above result.

If ζ be an operator defined by $\zeta H_n = (n+1)H_{n+1}$, and

$$\therefore \quad \zeta^p H_n = (n+1)\ldots(n+p)H_{n+p},$$

we have $$\nabla H_n = \zeta H_n - H_1H_n,$$

$$\nabla^2 H_n = \zeta^2 H_n - 2H_1\zeta H_n + 2(H_1^2 - H_2)H_n, \quad \&c.$$

Assuming $$\nabla^r H_n = \zeta^r H_n - rH_1\zeta^{r-1}H_n + r(r-1)(H_1^2 - H_2)\zeta^{r-2}H_n$$
$$+ r(r-1)(r-2)(2H_1H_2 - H_1^3 - H_3)\zeta^{r-3}H_n$$
$$+ r(r-1)(r-2)(r-3)(H_2^2 - 3H_1^2H_2 + 2H_1H_3 - H_4 + H_1^4)\zeta^{r-4}H_n + \&c.,$$

and operating with ∇ or $\zeta - H_1$, we find that this law holds also for $r+1$; therefore, &c. It is easily seen that, if $\phi(H)$ be the coefficient of $\zeta^m H_n$, that of $\zeta^{m-1}H_n$ will be $(\nabla - H_1)\phi(H)$ in form.

Putting $r = n$ and replacing the values of $\zeta^n H_n$, &c., we have $\nabla^n H_n$.

13185. (C. BICKERDIKE.)—C is a fixed and D a variable point on a circle whose centre is E. Produce CD to F, so that DF equals the diameter of the circle. Find the radius of curvature of the locus of F.

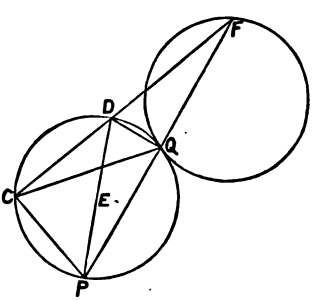
Solution by H. W. CURJEL, M.A.; Rev. T. WIGGINS, B.A.; *and others.*

Join F to the point P diametrically opposite to D, cutting the circle in Q.

Then DF = DP, and DQP is a right angle; $\therefore$ FQ = QP.

Hence the circle through F touching the given circle at Q is equal to the given circle, and we see that the locus of F may be described by the rolling of this circle on the given circle, and FQ is the normal at F.

Also, since the radii are equal, when EQ describes a small angle θ, FQ is rotated through an angle $\frac{1}{2}\theta$;



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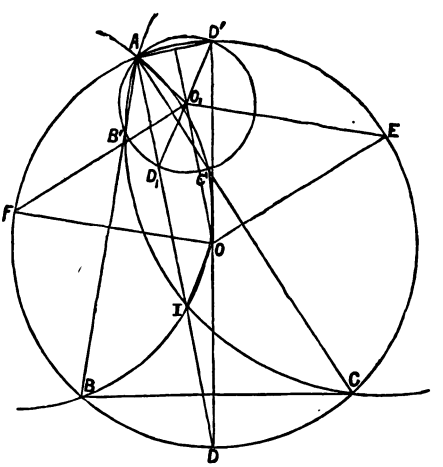
$\therefore$ if Q′, P′ are the new positions of Q and P, arc PP′ = 2 arc QQ′;
$\therefore$ centre of curvature at F is at R in QP, such that RP = 2RQ;
$\therefore$ radius of curvature = $\frac{2}{3}$FP. Locus of F is a cardioide.

13191. (Professor NEUBERG).—Soient O le centre du cercle circonscrit au triangle ABC, et I, I_a, I_b, I_c les centres du cercle inscrit et des cercles exinscrits. On prend sur les côtés BA, CA, vers le sommet A, les longueurs BB′ = CC′ = BC, et aussi, dans la direction opposée, BB″ = CC″ = BC. Cela posé: (1) Les droites B′C′, B″C″, B′C″, B″C′ sont respectivement perpendiculaires aux droites OI, OI_a, OI_b, OI_c; (2) les rayons des cercles circonscrits aux triangles AB′C′, AB″C″, AB″C′ sont respectivement égaux à OI, OI_a, OI_b, OI_c.

Solution by H. W. CURJEL, M.A.; W. E. JEFFARES, B.A.; *and others.*

Let the internal and external bisectors of angles A, B, C of triangle ABC cut the circumcircle in D, D′, E, E′, F, F′. Then E, F, E′, F′ are centres of circles

$AB'IC'I_b$, $AC'IB'I_c$,
$ACI_aB''I_c$, $ABI_aC''I_b$.



Let the perpendiculars from E and F on AB, AC cut in O_1, and from E′, F′ in O_2, and let O_1F, O_2E' and O_1E, O_2F' cut in O_3 and O_4. Then O_1, O_2, O_3, O_4 are clearly centres of circles AB′C′, AB″C″, AB″C′, AB′C″. And $EOFO_1$ is a rhombus, as also are $E'OF'O_2$, OFO_3E', $OF'O_4E$, and EF is perpendicular to AD; therefore O_1O, OO_2 are parallel to AD and OO_3, OO_4 to AD′.

$\therefore$ O_1OO_2 and O_3OO_4 bisect AD′, AD at right angles.

$\therefore$ D′ is on $\odot$s AB′C′, AB″C″ and D on $\odot$s AB″C′, AB′C″.

$\therefore$ if AI cuts $\odot$s AB′C′, AB″C″ in D_1, D_2,

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and AI_b cuts $\odot$s $AB''C'$, $AB'C''$ in D_3, D_4,

$D'D_1$, $D'D_2$, DD_3, DD_4 pass through O_1, O_2, O_3, O_4 respectively.

Now EF, E'F', E'F, EF' bisect AI and O_1O, AI_a and OO_2, AI_c and OO_3, AI_b and OO_4 at right angles.

$\therefore$ AO_1, AO_2, AO_3, $AO_4 = OI$, OI_a, OI_c, OI_b respectively,

and $\angle$s OIA, OI_aA, OI_cA, $OI_bA = \angle$s O_1AD_1, O_2AD_2, O_3AD_3, O_4AD_4
$= \angle$s O_1D_1A, O_2D_2A, O_3D_3A, O_4D_4A respectively.

$\therefore$ OI, OI_a, OI_c, OI_b are parallel to O_1D_1, O_2D_2, O_3D_3, O_4D_4

and $\therefore$ perpendicular to $B'C'$, $B''C''$, $B''C'$, $B'C''$ respectively.

13184. (P. W. Flood.)—In a circle 100 yards in diameter, three equal circles are inscribed touching each other and the outer circles. Find the diameter of a circle inscribed in the space at the centre.

Solution by I. Arnold, Professor Krishnamacharry, *and others.*

Let A, B, C be the centres of the three equal circles, O the centre of the circumscribing circle, and D the point of contact of the circle A.

Join OB, and produce OA to D.

Now, in $\triangle AOB$,

$\angle AOB = 120°$

and $\angle OBA = 30°$.

Let γ = the radius of the equal circles; then we have

$\sin 120° : \sin 30° :: 2\gamma : AO$;

$\therefore$ $AO = \frac{2\gamma}{\sqrt{3}}$ and $\gamma + \frac{2\gamma}{\sqrt{3}} = 50$.

Hence $\sqrt{3}\gamma + 2\gamma = 50\sqrt{3}$

and $$\gamma = \frac{50\sqrt{3}}{2+\sqrt{3}} = \frac{86{\cdot}6}{3{\cdot}732} = 23{\cdot}2;$$

$\therefore$ diameter of A = 46·4, and OG or OE the radius of circle inscribed in space between the three equal circles

$= 50 - 2\gamma = 50 - 46{\cdot}4 = 3{\cdot}6$ yards,

and its diameter $= 7{\cdot}2$ yards.

[Mr. Curjel gives the solution, briefly, thus:—

Let r be the radius of the given circle.

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Then, radius of each of the three equal circles = radius of in-circle of triangle (sides $2r$, $2r$, $2r\sqrt{3}$) $= \dfrac{r\sqrt{3}}{2+\sqrt{3}}$;

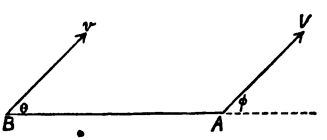
$$\therefore \quad \text{radius of small circle} = r - \frac{2r\sqrt{3}}{2+\sqrt{3}};$$

$$\therefore \quad \text{diameter} = 2r(2-\sqrt{3})^2 = 7{\cdot}18 \text{ yards.}]$$

13028. (Rev. T. Wood Robinson, R.N.)—Two ships A and B are at sea; if A be faster than B, then, as long as A keeps B on a certain fixed bearing, (1) B can never close on A, (2) if B tries to close by keeping A dead ahead, show their distance apart will remain constant, and (3) find their paths in this case.

Solution by the Proposer, Professor Mukhopadhyay, *and others.*

Let velocities of A and B be V and v, and their directions to AB be θ and ϕ.



Resolving along BA, we get velocity of A relative to B in direction BA

$$= \mathrm{V}\cos\phi - v\cos\theta.$$

B cannot close on A if this is a positive;

i.e., if $\mathrm{V}\cos\phi \not< v\cos\theta$; *i.e.*, if $\mathrm{V}\cos\phi \not< v$, whatever θ be;

i.e., if $\cos\phi \not< v/\mathrm{V}$.

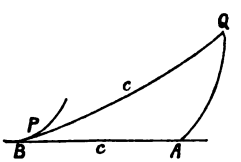
(2) $\theta = 0$; therefore, if $\cos\phi = v/\mathrm{V}$,

A's velocity relative to B along BA = 0;

i.e., their distance apart remains constant.

(3) Let ξ and η be coordinates of P, and x and y be coordinates of Q.

If θ be inclination of PQ to BA,



$$\frac{d\xi}{dt} = v\cos\theta, \qquad \frac{d\eta}{dt} = v\sin\theta,$$

$$x = \xi + c\cos\theta, \qquad y = \eta + c\sin\theta,$$

$$\mathrm{V}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d\xi}{dt} - c\sin\theta\frac{d\theta}{dt}\right)^2 + \left(\frac{d\eta}{dt} + c\cos\theta\frac{d\theta}{dt}\right)^2$$

$$= \left(v\cos\theta - c\sin\theta\frac{d\theta}{dt}\right)^2 + \left(v\sin\theta + c\cos\theta\frac{d\theta}{dt}\right)^2 = v^2 + c^2\left(\frac{d\theta}{dt}\right)^2;$$

$$\therefore \quad \left(\frac{d\theta}{dt}\right) = \frac{(\mathrm{V}^2 - v^2)^{\frac{1}{2}}}{c};$$

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$$\therefore\quad \theta = \frac{(V^2-v^2)^{\frac{1}{2}}}{c}t, \quad \text{since} \quad \theta = 0 \quad \text{when} \quad t = 0;$$

$$\therefore\quad \frac{d\xi}{dt} = v\cos\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t;$$

$$\therefore\quad \xi = -\frac{cv}{(V^2-v^2)^{\frac{1}{2}}}\sin\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t$$

and
$$\eta = \frac{cv}{(V^2-v^2)^{\frac{1}{2}}}\left\{\cos\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t-1\right\}.$$

Therefore B describes a circle.

Again,
$$x = c\cos\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t - \frac{cv}{(V^2-v^2)^{\frac{1}{2}}}\sin\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t$$

$$y = c\sin\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t + \frac{cv}{(V^2-v^2)^{\frac{1}{2}}}\left\{\cos\frac{(V^2-v^2)^{\frac{1}{2}}}{c}t-1\right\}.$$

Therefore A also describes a circle concentric with circle described by B.

13134. (Professor Zerr.)—A man raises 1 chicken the first year, 6 the second, 35 the third, 180 the fourth, 921 the fifth, 4,626 the sixth, 23,215 the seventh, 116,160 the eighth, and so on. How many does he raise the twentieth year, and how many in the 20 years?

Solution by Rev. J. Cullen, B.A.; Professor Nath Coondoo; *and others.*

We have $6 = 5\times1+1$, $35 = 5\times6+5$, $180 = 5\times35+5$,
$921 = 5\times180+21$, &c.

Hence, let u_n be the nth term of $1+35+921+$ &c.,
w_n ,, ,, $6+180+4626+$ &c.,
v_n ,, ,, $1+5+21+85+$ &c.;

$\therefore\quad v_{n+1} = 4v_n+1$ or $v_n = (4^n-1)/3$, $u_{n+1} = 25u_n+3(4^n-1)+1$,
$w_{n+1} = 25w_n+2(4^{n+1}-1)$.

Hence $u_n = \{5\times25^n+7-3\times4^{n+1}\}/84$, $w_n = \{25^{n+1}+7-2\times4^{n+2}\}/84$.

The number raised in the twentieth year is w_{10}, or
$\{25^{11}+7-2\times4^{12}\}/84 = 28,383,163,779,300.$

In 20 years the number is $\Sigma(u_n+w_n)$ to 10 terms, or
$\{750(25^{10}-1)/24+140-44(4^{11}-4)/3\}/84 = 35,478,954,491,110.$

7384. (By Professor S. Réalis.)—Etant donnée la série illimitée 7, 13, 25, 43, 67, 97, 133, 175, ..., dont le terme général, celui qui en a n avant lui, est $xn = 3(n^2+n)+7$, démontrer les propositions suivantes:—

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(1) sur cinq termes consécutifs, pris à volonté dans la série, un terme est divisible par 5; (2) sur sept termes consécutifs, deux sont divisible par 7; (3) sur treize termes consécutifs, deux sont divisible par 13; (4) aucun terme de la série n'est égal à un cube; (5) une infinité de termes, tels que $x_2 = 25$, $x_{37} = 4225$, &c., sont des carrés divisible par 25; (6) la deuxième et la troisième proposition sont comprises, comme cas particuliers, dans la suivante: si de N est un nombre premier, de la forme $6n+1$, sur N termes consécutifs de la série, deux sont divisible par N; (7) on peut affirmer aussi que, à l'exception de 5, aucun nombre premier de la forme $6m-1$ ne peut diviser la série.

Solution by H. W. Curjel, M.A.; Prof. Swaminatha Aiyar; *and others.*

$x_n = \frac{1}{4}\{3(2n+1)^2+25\}$. One of any five consecutive odd numbers is divisible by 5. Hence (1).

If $6m+1 = N$ is a prime,

$$\left(\frac{-1}{6m+1}\right) = (-1)^m, \quad \left(\frac{3}{6m+1}\right)\left(\frac{6m+1}{3}\right) = (-1)^{\frac{1}{2}(2\times 6m)} = (-1)^m,$$

$$\left(\frac{6m+1}{3}\right) = \left(\frac{1}{3}\right) = 1; \quad \therefore \left(\frac{-3}{6m+1}\right) = 1,$$

i.e., -3 is a quadratic residue of N.

Therefore $3y^2+25 \equiv 0 \mod N$ is possible, and will give two values of $y < N$, and therefore two odd values of y in every series of N consecutive odd numbers.

Hence two of any N consecutive terms of the series are divisible by N. Hence (6), (2), (3).

If $6m-1 = N$ is a prime,

$$\left(\frac{-3}{6m-1}\right) = \left(\frac{-1}{6m-1}\right); \quad \left(\frac{-1}{6m-1}\right) = (-1)^{3m-1};$$

$$\left(\frac{3}{6m-1}\right)\left(\frac{6m-1}{3}\right) = (-1)^{\frac{1}{2}[2(6m-2)]} = (-1)^{3m-1};$$

$$\left(\frac{6m-1}{3}\right) = \left(\frac{-1}{3}\right) = -1; \quad \therefore \left(\frac{-3}{6m-1}\right) = -1,$$

i.e., -3 is a non-residue of N. Hence (7).

If x_n is a cube, it is the cube of a number of the form $6m+1$, and x_n-1 is divisible by 9, which is impossible since $n^2+n+2 = n^2-1+n+3$ cannot be divisible by 3. Hence (4).

Again, $x-3y^2 = 1$ has an infinite number of solutions; hence (5).

12855. (Professor Zerr.)—One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometres per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 100 grams mass, and moving with a velocity each of 10 kilometres per second; collisions take place, and finally, after a number of encounters, the average kinetic

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energy of each of the two thousand balls is the same. Show that this is $2{\cdot}75\ (10)^{11}$ in the centimetre-gram-system.

Solution by Rev. J. L. KITCHIN, Professor GOPALACHANAR, *and others.*

Let K = average K. Energy of the 2000 balls.

$\therefore$ $2000\,K$ = K.E. of the 10-gram balls + K.E. of 100-gram balls.

$= \frac{1}{2} \times 10 \times 1000 \times (10^5)^2.$

$+ \frac{1}{2} \times 100 \times 1000 \times (10^5)^2.$

$= \frac{1}{2} \times 10^3 \times 10^{10} \times 110.$

$\therefore$ $K = \dfrac{11 \times 10^{11}}{4} = 2{\cdot}75 \times 10^{11}.$ [C.G.S. system.]

13183. (A. GORDON, M.A.)—Two parallel planes are drawn in an ellipsoid, each parallel to one of the principal sections. Two lines are described with these sections as bases and the centre of the ellipsoid as common vertex. Required the relation between x, the distance apart of the planes, and V, the volume of the ellipsoid contained between the cones ($x \propto V$).

Solution by H. W. CURJEL, M.A.; Professor GOPALACHANAR; *and others.*

Let the parallel planes be perpendicular to the axis AOA′ ($=2a$), the other axes being BOB′, COC′ ($=2b,\ 2c$).

Deform the ellipsoid into a sphere by compressing or stretching along the axes BOB′, COC′; then volumes are altered in the ratio a^2/bc.

Area of sphere between the planes $= 2\pi ax$.

$\therefore$ Volume between the cones $= \frac{2}{3}\pi a^2 x$.

$\therefore$ in case of ellipsoid, volume $= \frac{2}{3}\pi bcx$.

13075. (Professor ZERR.)—I have a circular-section basin 12 inches in perpendicular height; the diameters are as follows:—at base, 2 inches; 1-inch perpendicular height, 6 inches; 2-inches perpendicular height, 18 inches; 3-inches perpendicular height, 54 inches; and so on, the diameter being trebled for every inch in height. After rain, the water in the basin is 6 inches deep; what was the rainfall?

Solution by Rev. J. CULLEN, B.A.; Prof. KRISHMACHANDRA DE; *and others.*

The generating curve is $y = 3^x$;

$$\therefore \text{ vol.} = \pi \int_0^6 3^{2x}\,dx = \pi\,(3^{12}-1)/2\log_e 3.$$

Area at mouth $= \pi\, 3^{24}$;

$\therefore$ rainfall $= (3^{12}-1)/(2 \times 3^{24} \log_e 3) = {\cdot}000000856$ inch.

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7589. (By W. Perryman.)—Show that

$$mn = \sin(m\sin n) + \tfrac{1}{2}\cdot\tfrac{1}{3}\sin^3(m\sin n) + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}\sin^5(m\sin n) + \ldots$$
$$+ \sin(\tfrac{1}{2}\cdot\tfrac{1}{3}m\sin^3 n) + \tfrac{1}{2}\cdot\tfrac{1}{3}\sin^3(\tfrac{1}{2}\cdot\tfrac{1}{3}m\sin^3 n) + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}\sin^5(\tfrac{1}{2}\cdot\tfrac{1}{3}m\sin^3 n) + \ldots$$
$$+ \sin(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}m\sin^5 n) + \tfrac{1}{2}\cdot\tfrac{1}{3}\sin^3(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}m\sin^5 n)$$
$$+ \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}\sin^5(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}m\sin^5 n) + \ldots$$

Solution by H. W. Curjel, M.A.; Professor Mukhopadhyay; *and others.*

The double series is clearly absolutely convergent; hence, from expansion of $\sin^{-1}x$, we see, by summing the rows, that series

$$= m\sin n + \tfrac{1}{2}\cdot\tfrac{1}{3}m\sin^3 n + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}m\sin^5 n + \&c.$$
$$= mn \text{ if } m\sin n \text{ and } n \text{ lie between } \tfrac{1}{2}\pi \text{ and } -\tfrac{1}{2}\pi.$$

9810. (W. J. C. Sharp, M.A.)—A, B, C are fixed points, P another point; construct the resultant of forces acting along PA, PB, PC, when they are proportional to

(1) PA^2, PB^2, and PC^2; (2) $\frac{1}{PA}$, $\frac{1}{PB}$, and $\frac{1}{PC}$;

(3) $\frac{1}{PA^2}$, $\frac{1}{PB^2}$, and $\frac{1}{PC^2}$.

Solution by H. J. Woodall, A.R.C.S.; Professor Sarkar; *and others.*

(1) It will be necessary to find lines in the given ratios; this is easily done by means of the proposition in Euclid VI., 11.

(2) The reciprocals of given lines may be obtained by means of Euclid III., 35.

(3) This results from successive application of Euclid VI., 11, and Euclid III., 35.

When the lines have been obtained there is no difficulty in finding the resultant. Thus, if G be centre of gravity of triangle A'B'C', 3PG is the resultant.

12712. (Professor Symmond.)—Show that, of all curves of a given length, traced on one plane between two given points, and made to revolve around a common axis situated in that plane, the catenary generates a minimum area.

Solution by Professors Colaw, Mukhopadhyay, *and others.*

Represent the given length by L; then according to the notation of the calculus of variations, we require a minimum value of

$$\int_z^w V\,dx = \int_z^w \left[2\pi y\,(1+p^2)^{\frac{1}{2}} + L\right] dx \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

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in which $z = x_0$ and $w = x_1$. Transforming the equation

$$V = Pp + C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

by means of (1), we have

$$2\pi y(1+p^2)^{\frac{1}{2}} + L = 2\pi y p^2/(1+p^2)^{\frac{1}{2}} + C \quad \ldots\ldots\ldots\ldots\ldots (3).$$

$$\therefore \quad y/(1+p^2)^{\frac{1}{2}}, \quad = (C-L)/2\pi, \quad = a \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

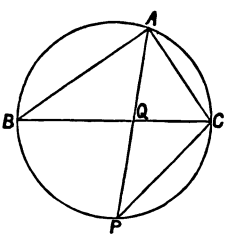
which is the differential equation of the required curve. Remembering that p represents dy/dx, we have, from (4),

$$X = a\int \frac{dy}{(y^2-a^2)^{\frac{1}{2}}} = C' \log\left[y + (y^2-a^2)^{\frac{1}{2}}\right] \quad \ldots\ldots\ldots\ldots (5),$$

which is an equation of the catenary.

13203. (Editor.)—If, in a triangle ABC, right-angled at A, the bisector of the angle A meet the side a (*i.e.*, BC) in Q and the circum-circle in P, prove that $PA : PQ = (b+c)^2 : a^2$.

Solution by W. E. Jeffares, B.A.; Rev. T. Wiggins, B.A.; *and others.*



If we produce BA to D, so that AD = AC,

then $$\frac{PA}{PQ} = \frac{PA^2}{PB^2} = \frac{PA^2}{PC^2} = \frac{BD^2}{BC^2}$$

(since $\triangle$s APC and DBC are equiangular)

$$= \frac{(b+c)^2}{a^2}; \quad \therefore \quad \frac{PA}{PQ} = \frac{(b+c)^2}{a^2}.$$

13010. (Professor Shields.)—Four tortoises, A, B, C, D, start from the same place, on a level plain. B and C travelled at the same rate; the travelling rates of A and B were as 17 to 18 respectively, and the travelling rates of C and D were as 18 to $19\frac{1}{17}$ respectively; or the rates of all were as 17 to 18 to $19\frac{1}{17}$ respectively. A could travel one mile in 1 hr. 26 min. 24 sec., and A travelled due west a certain distance, thence due north a certain distance. B travelled due north the cube of A's distance west *plus* his distance west; thence B travelled due east the fifth power of A's distance north, *plus* his distance north; C travelled due east a certain distance; thence C travelled due south a certain distance, and D travelled due south the cube of C's distance east *plus* his distance east; thence D travelled due west the fifth power of C's distance south, *plus* his distance south, when it was found that the sum of the distances B and D travelled north and south was 467280 inches, and the sum of the

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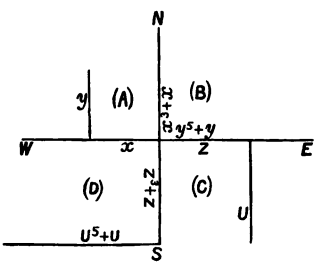
distances B and D travelled east and west was 1949266638 inches, and that the product of the distance A and C travelled east and west *plus* the square of the difference of these distances *plus* 1, was 4720, and that the product of the distances A and C travelled north and south *plus* the square of the difference, squared, *plus* the product multiplied by the square of the difference, *plus* 1, was 19689562. Find how far each travelled in all, and how long it required for B and D each to make the entire trip from starting place to the end.

Solution by Rev. J. L. KITCHIN, Professor KRISHNAMACHARRY, *and others*.

The diagram gives a clear statement of the conditions of the problem; then



$$x^3+z^3+x+z = 467280 \quad \ldots (1).$$

$$U^5+y^5+U+y = 1949266638 \ldots (2).$$

$$xz+(x-z)^2+1 = 4720 \ldots\ldots (3).$$

$$\{Uy+(U-y)^2\}^2+Uy\,(U-y)^2+1 = 19689562 \ldots\ldots (4).$$

(1) becomes $(x+z)(x^2-xz+y^2+1) = 467280 = (x+z)\times(3)$;

$$\therefore \quad x+z = \frac{46728}{472} = 99.$$

Also $x-z = \pm 55$; $\therefore$ $x = 77$ or 22, $z = 22$ or 77.

Again, (2) becomes $(U+y)\{1+U^4-U^3y+U^2y^2-Uy^3+y^4\} = 1949266638$.

The part between $\{\ \}$ is the same as (4);

$$\therefore \quad U+y = 1949266638/19689562 = 99.$$

Using this in (4), we obtain a quadratic in Uy, giving $Uy = 1944$ or 7857. The latter gives impossible values of U and y.

From $U+y = 99$, $Uy = 1944$; $\therefore$ $U-y = \pm 45$;

$$\therefore \quad U = 72, 27; \quad y = 27, 72.$$

A travels $\frac{44}{14}$ feet per sec.; $\therefore$ B $\frac{44}{11}$, C $\frac{44}{11}$, and D $\frac{330}{330}$ feet per sec. each. Take first values of x, y, U, z, and we get the values of the trips of B and D in 1 sec.

13215. (H. W. CURJEL, M.A.)—Prove that, n, p, q being any positive quantities,

$$\frac{1}{\Gamma(p+q)} - n\frac{q}{\Gamma(p+q+1)} + \frac{n(n-1)}{1.2}\frac{q(q+1)}{\Gamma(p+q+2)} - \frac{n(n-1)(n-2)}{1.2.3}\frac{q(q+1)(q+2)}{\Gamma(p+q+3)} + \ldots = \frac{\Gamma(p+n)}{\Gamma(p)\,\Gamma(p+q+n)}.$$

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Solution by Professor CHAKRAVARTI, *the* PROPOSER, *and others.*

$$\frac{\Gamma(q)\,\Gamma(p+n)}{\Gamma(p+q+n)} = \int_0^1 x^{q-1}(1-x)^{p+n-1}\,dx$$

$$= \int_0^1 \left(x^{q-1} - nx^q + \frac{n(n-1)}{1.2}x^{q+1} - \ldots \right)(1-x)^{p-1}\,dx \text{ if } n \text{ is positive}$$

$$= \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} - \frac{n}{1}\frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)} + \frac{n(n-1)}{1.2}\frac{\Gamma(p)\,\Gamma(q+2)}{\Gamma(p+q+2)} - \&c.;$$

therefore

$$\frac{\Gamma(p+n)}{\Gamma(p)\,\Gamma(p+q+n)} = \frac{1}{\Gamma(p+q)} - \frac{n}{1}\frac{q}{\Gamma(p+q+1)} + \frac{n(n-1)}{1.2}\frac{q(q+1)}{\Gamma(p+q+2)} - \&c.$$

This may be written

$$\frac{\Gamma(p+n)\,\Gamma(p+q)}{\Gamma(p)(p+q+n)} = 1 - \frac{n}{1}\frac{p}{p+q} + \frac{n(n-1)}{1.2}\frac{q(q+1)}{(p+q)(p+q+1)} - \&c.$$

or

$$\frac{\Gamma(\gamma-\beta-\alpha)\,\Gamma(\gamma)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\alpha)} = 1 + \frac{\alpha}{1}\frac{\beta}{\gamma} + \frac{\alpha(\alpha+1)}{1.2}\frac{\beta(\beta+1)}{\gamma(\gamma+1)} + \ldots,$$

where $\gamma-\beta$, $\gamma-\alpha$, and $-\alpha$ are positive: this may be easily shown to be true in all cases where the series on the right is convergent, *i.e.*, where $\gamma-\alpha-\beta > 0$. (This result is arrived at in a different manner in FORSYTH'S *Differential Equations*, §§ 125, 126.)

[Mr. CULLEN gives the solution thus:—

$$\int_0^1 x^{q-1}(1-x)^{p+n-1}\,dx = \int_0^1 x^{q-1}(1-x)^{p-1}(1-x)^n\,dx$$

$$= \int_0^1 x^{q-1}(1-x)^{p-1}\,dx - n\int_0^1 x^q(1-x)^{p-1}\,dx$$

$$+ \frac{n(n-1)}{1.2}\int_0^1 x^{q+1}(1-x)^{p-1}\,dx - \&c.$$

Expressing these integrals as Γ functions and employing the property $\Gamma(q+1) = q\Gamma(q)$, we have the required equation on dividing across by $\Gamma(p)\,\Gamma(q)$.]

13188. (Rev. T. ROACH, M.A.)—Prove geometrically that

$$\cos^{-1}\tfrac{8}{17} + \sin^{-1}\tfrac{5}{13} = \cos^{-1}\tfrac{33}{221}.$$

Solution by R. CHARTRES, *the* PROPOSER, *and others.*

Let $BOA = \sin^{-1}\frac{5}{13}$, $COB = \cos^{-1}\frac{8}{17}$.

Take $OB = OC = 17 \times 13$.

Draw perpendiculars BA, CM, CK, KN, KR.

Then $BA = 5 \times 17$ and $OK = 8 \times 13$;

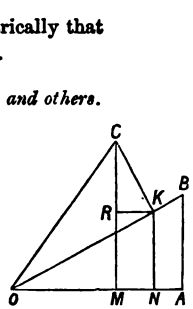
$\therefore$ $CK = 15 \times 13$.

Now $RCK = BOA$. $\therefore$ $RK : CK = 5 : 13$.

$\therefore$ $RK = 75$, $ON : OK = OA : OB = 12 : 13$.

$\therefore$ $ON = 96$, $OM = 21$;

$\therefore$ $\cos^{-1}\frac{8}{17} + \sin^{-1}\frac{5}{13} = COB + BOA = \cos^{-1} OM/OC = \cos^{-1}\frac{33}{221}$.



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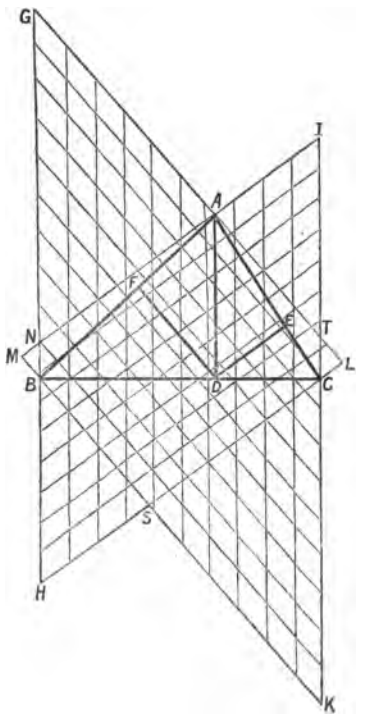
13254. (Professor RAMASWAMI AIYAR.)—ABC is a given acute-angled triangle. Points P, Q, R are taken at random in the sides BC, CA, AB; and perpendiculars are drawn at P, Q, R to those sides. Show that the probability that the triangle formed by these perpendiculars lies entirely within the triangle ABC is $\frac{1}{4}(\sin A \sin B \sin C)/(\cot A + \cot B + \cot C)$.

Solution by D. BIDDLE; H. W. CURJEL, M.A.; *and others.*

Let n perpendiculars be drawn to each side of ABC. Let k, l, m be intersections, respectively, between P- and Q-, P- and R-, Q- and R-perpendiculars.

There will be n^2k-, n^2l-, and n^2m-intersections in all, and each intersection will be the apex of n triangles of varying size, the total number of triangles formed by the perpendiculars being n^3, all of which will lie within the irregular space represented in the diagram by AGNMBHSKCLTIA,

$$= \tfrac{1}{2}\{(a+b)\cot C + (a+c)\cot B + (b+c)\cot A\}.$$



But uniform distribution of the three sets of intersections does not extend beyond the limited area

ANBSCTA, = 2ABC.

The n^2k-, n^2l-, n^2m-intersections are restricted to the respective parallelograms

CHNI, BGTK, ALSM.

Now, taking AD perpendicular to BC, the following propositions appear valid, if n be supposed infinite.

1. AD is crossed by AD/CI of the perpendiculars to AC, and by AD/BG of the perpendiculars to AB, and these values (equivalent, respectively, to AE/AC and AF/AB) are proportioned to unity as $\sin^2 C$ and $\sin^2 B$.

2. The length of perpendicular within ABC varies from 0 to 2 of the average (which, on BC, is reached when P is midway from B to D or from C to D); and there are within the triangle $\frac{1}{4}\sin^2 C$, $\frac{1}{4}\sin^2 B$, $\frac{1}{4}\sin^2 A$ of the total k-, l-, m-intersections, respectively.

3. Only a portion of the two kinds of perpendiculars which cross a given intercept (such as AD) intersect each other within ABC. But the

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area represented by AEDF contains all such mutual ntersections as in the particular instance do occur.

4. The mutual intersections referred to are all those of the m-kind within the special area, for no intersections of this kind can occur in AEDF except of perpendiculars having their feet in AE and AF. Each such m-intersection counts for one triangle.

5. In the said area, as in the whole triangle, the k-, l-, m-intersections are as $\sin^2 C$, $\sin^2 B$, $\sin^2 A$, and they are distinct; at least, in the smallest element of the area their ratio is the same. Thus the m-intersections are, in any such given area, as $\sin^2 A/(\sin^2 A + \sin^2 B + \sin^2 C)$ of the whole mass of intersections occurring within that area.

Consequently, we have

$$\int_0^2 \frac{\sin^2 B \cdot \sin^2 C}{4} \cdot \frac{\sin^2 A}{\sin^2 A + \sin^2 B + \sin^2 C} x^2 \cdot dx$$

$$= \tfrac{2}{3} \cdot \frac{\sin^2 A \cdot \sin^2 B \cdot \sin^2 C}{\sin^2 A + \sin^2 B + \sin^2 C} = \tfrac{1}{3} \cdot \frac{\sin A \cdot \sin B \cdot \sin C}{\cot A + \cot B + \cot C}.$$

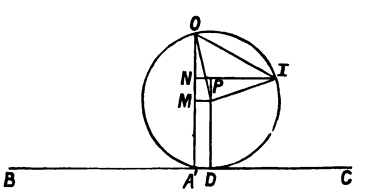
In an equilateral triangle the probability is $\frac{1}{8}$.

3706. (R. Tucker, M.A.)—If d, D be the diameters of the inscribed and circumscribed circles of a triangle, δ the distance between their centres, and h the harmonic mean between the radii of the two circles which can be described through the above centres touching a side of the triangle, prove that $\Sigma\left(\frac{1}{h-d}\right) = \frac{4d+D}{\delta^2-d^2}$.

Solution by Professors Sanjána, Krishnamacharry, *and others.*

Let BC be a side of the triangle, O the circumcentre, I the incentre, and P the centre of one circle touching BC and passing through O and I. Draw perpendiculars as in the diagram; let

$OA' = k_1$, $NA' = r$ (inradius), $IN = q$, $PD = p$, $PM = x$.



Then $\delta^2 = OI^2 = q^2 + (k_1 - r)^2 = q^2 + r^2 + k_1^2 - 2rk_1$;

$p^2 = PO^2 = x^2 + (k_1 - p)^2$, so that $x^2 - 2k_1 p + k_1^2 = 0$;

and $p^2 = PI^2 = (q-x)^2 + (r-p)^2$, so that $x^2 - 2qx + q^2 + r^2 - 2rp = 0$.

From the last two equations by subtraction we get

$$x = (2k_1 p + q^2 + r^2 - k_1^2 - 2rp)/2q;$$

substituting in the first of these, we get

$$\{q^2 + r^2 - k_1^2 + 2p(k_1 - r)\}^2 - 8q^2 k_1 p + 4q^2 k_1^2 = 0,$$

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a quadratic to determine p. This reduces to

$$4p^2(k_1-r)^2-4p(k_1+r)(q^2+r^2+k_1^2-2k_1r) + (q^2+r^2+k_1^2-2k_1r)(q^2+r^2+k_1^2+2k_1r) = 0,$$

that is, $$4p^2(k_1-r)^2-4p(k_1+r)\delta^2+\delta^2(q^2+r^2+k_1^2+2k_1r) = 0.$$

If h_1 be the harmonic mean between the two values of p,

$$h_1 = \frac{2p_1p_2}{p_1+p_2} = \frac{2\delta^2(q^2+r^2+k_1^2+2k_1r)}{4\delta^2(k_1+r)} = \frac{\delta^2+4k_1r}{2(k_1+r)};$$

$$\therefore\quad h_1-d = \frac{\delta^2+2k_1d}{2k_1+d}-d = \frac{\delta^2-d^2}{2k_1+d}.$$

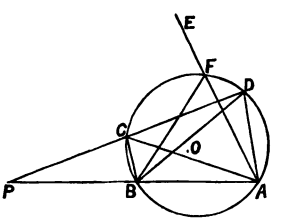
Thus $$\Sigma\left(\frac{1}{h_1-d}\right) = \frac{2(k_1+k_2+k_3)+3d}{\delta^2-d^2} = \frac{2R(\cos A+\cos B+\cos C)+3d}{\delta^2-d^2}$$

$$= \frac{2R+8R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C+3d}{\delta^2-d^2} = \frac{D+4d}{\delta^2-d^2}.$$

13231. (Professor Finkel.)—A pole, a certain length of whose top is painted white, is standing on the side of a hill. A person at A observes that the white part of the pole subtends an angle equal to a, and on walking to B, a distance a, directly down the hill towards the foot of the pole, the white part subtends the same angle. What is the length of the white part, if the point B is at a distance b from the foot of the pole?

Solution by W. E. Heal, M.A.; W. E. Jeffares, B.A.; *and others.*

Describe a circle O through A, B, and at A lay off the angle BAE equal to a, cutting the circle O in F. BF is the required length.



For, let BF = l, and x a root of

$$x(x+l) = b(b+a).$$

With P as centre, and

radius = x = PC, or $x+l$ = PD,

describe a circle cutting O in C or D.

Join PC or PD, and CD = l = BF satisfies the condition of the question.

The result is, of course, indeterminate. The Proposer probably intended the angle made by the pole with the side of the hill to be a given quantity. But, if the angle at P were given, the circle O would have to be of determined radius.

13243. (R. Knowles, B.A.)—From a point T in the directrix of a parabola, whose focus is F, and vertex A, tangents are drawn to meet

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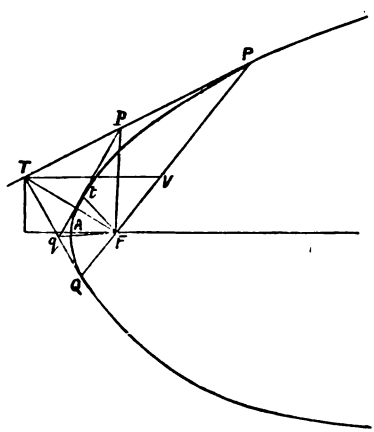
the curve in P, Q; show that (1) $TP \cdot TQ = (FT)^3/AF = \sqrt{(AF \cdot PQ^3)}$; (2) if a third tangent at t meets these in p, q, $(pq)^2 = PQ \cdot Ft$.

Solution by H. W. Curjel, M.A.; W. J. Greenstreet, M.A.; *and others.*

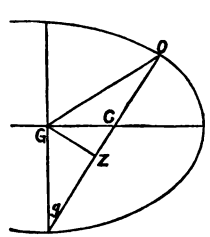


Bisect PQ in V.

$$2AF : TF = TF : TV$$
$$= TF : \tfrac{1}{2}PQ;$$
$$\therefore \quad FT^2 = AF \cdot PQ;$$
$$\therefore \quad (TP \cdot TQ)^2$$
$$= PQ^2 \cdot PF \cdot FQ$$
$$= PQ^2 \cdot FT^2$$
$$= FT^6/AF^2$$
$$= PQ^3 \cdot AF;$$
$$Ft \cdot PQ = Ft \cdot PF + Ft \cdot FQ$$
$$= Fp^2 + Fq^2$$
$$= pq^2.$$

13264. (W. E. K. Hare, B.A., B.C.S.)—(1) Prove that all chords of an ellipse which subtend a right angle at a fixed point O on the ellipse pass through a fixed point Z on the normal at O. (2) Determine the locus of Z for different positions of O on the ellipse.

Solution by Prof. Krishnamacharry; H. W. Curjel, M.A.; *and others.*



Let XX′, YY′, ... be chords subtending a right angle at O. Then the pencil O (XX′, YY′, ...) is in involution. Therefore XX′, YY′, ... meet in a point Z which is in normal at O, since, when X is at O, XX′ coincides with normal at O. The double lines of the involution O (XX′, YY′, ...) are the circulars of O, since the involution is orthogonal. Therefore Z is on the diameter through A, the reflexion of O with respect to the axis of the conic.

Hence, in the parabola, the axis bisects OZ. Therefore locus of Z is an equal parabola with the same axis, and the distance between the vertices = latus rectum. In central conic (centre C), let OZ meet axes in G, g.

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Then $$\frac{CZ}{CO} = \frac{GZ}{GO} = \frac{Gg}{2OG + Gg} \quad \text{[for } (gG, ZO) \text{ is harmonic]}$$

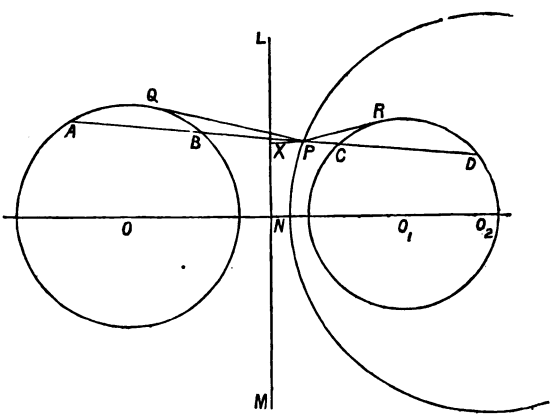
$$= \frac{Og - OG}{Og + OG} = \frac{(a/b) - (b/a)}{(a/b) + (b/a)} = \frac{a^2 - b^2}{a^2 + b^2}.$$

Therefore locus of Z is a similar, and similarly situated, concentric conic.

4259. (Elizabeth Blackwood.)—Two circles, which are given in magnitude and position, cut a straight line, which is also given in position, in A, B and C, D. Determine the position of a point P in the intercepted part BC of this line, so that, drawing the tangents PQ, PR, the ratio (PQ : PR) may be given.

Solution by W. E. Jeffares, B.A.; Prof. Swaminatha Aiyar; *and others.*

Draw LM the radical axis of the two circles O, O_1 and let $m : n$ be the given ratio. Divide OO_1 externally in O_2, so that $OO_2 : O_1O_2 :: m^2 : n^2$.



With O_2 as centre, describe a coaxal circle with the circles O, O_1, that is, having the (radius)^2 equal to $O_2N^2 - (ON^2 - Y^2)$, where Y^2 is the radius of circle O. Where this circle cuts the given line AD in P is the required point.

From P draw the tangents PQ and PR, and the perpendicular PX to the radical axis $PQ^2 = 2OO_2 \times PX$ [see Casey's *Sequel*, Sixth Edition, p. 113 (1)].

$PR^2 = 2O_1O_2 \times PX$; $\therefore$ $PQ^2/PR^2 = OO_2/O_1O_2 = m^2/n^2$, $PQ/PR = m/n$.

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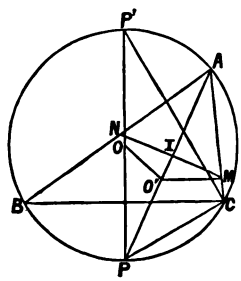
13126. (Rev. G. RICHARDSON, M.A.)—If a circle be drawn touching two sides of a triangle so that the chord of contact passes through the in-centre or one of the ex-centres, prove that the circle touches the circumcircle.

Solution by the PROPOSER, Professor COCHEZ, *and others.*

Let ABC be the triangle, I, O its in-centre and circumcentre; and let O′ be the centre of the circle which touches AB, AC in N, M.



Produce AIO′ to meet circumcircle in P, draw the diameter POP′, and join CP, CP′, OO′; then $R^2 - OO'^2 = PO'.\ O'A$

$$= (PI - O'I)\ O'A$$
$$= PC\ .\ O'A - O'I\ .\ O'A$$
$$= 2R\ .\ O'M - O'M^2;$$

from similar triangles PCP′, O′MA; and O′MI, O′AM: $\therefore\ R^2 - 2R\ .\ O'M + O'M^2 = OO'^2$

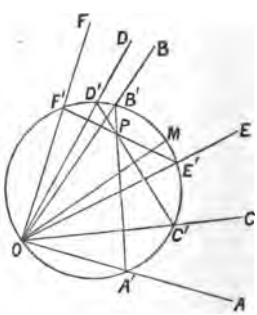
or $R - O'M = OO'$.

The proof is similar for the ex-centres.

13220. (Professor NEUBERG.)—On projette le point M en A′ et B′ sur les côtés de l'angle AOB, en C′ et D′ sur les côtés de l'angle CO′D, en E′ et F′ sur les côtés de l'angle EO′′F. Trouver (1) le lieu d'un point M; tel que les droites A′B′, C′D′, E′F′ concourent en un même point P; (2) lieu de ce point P.

Solution by H. W. CUBJEL, M.A.; Prof. MUKHOPADHYAY; *and others.*

On OM as diameter, describe a circle; it will pass through A′, B′, C′, D′, E′, F′. Hence the necessary and sufficient condition for A′B′, C′D′, E′F′ to be concurrent is that the pencil O (AB, CD, EF) should be in involution. If this condition is satisfied, M may be anywhere in the plane. If we move O along ⊙A′B′O so as to describe an arc subtending an $\angle\theta$ at the centre, and then turn the whole figure through an $\angle(-\theta)$ about the centre of the circle and through an $\angle\frac{1}{2}\theta$ about the new position of O, this will be equivalent to moving M along an arc of a circle with centre O. Hence, as M describes this circle, P describes an epitrochoid, O being the centre of the fixed circle, $\frac{1}{3}$OM its radius, and $\frac{1}{6}$OM the radius of the rolling circle. Hence P also may be anywhere in the plane.



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13212. (I. Arnold.)—ABCD is a quadrilateral, and AC, BD the diagonals intersecting in E. Make BF = DE, and AR = CE; and let Et_1 be $\frac{1}{3}$ of EF, and $Et = \frac{1}{3}$ of ER. Through t and t_1 draw ts and t_1s parallel to the diagonals; and prove that s is the centre of gravity of the quadrilateral.

Solution by H. W. Curjel, M.A.; H. B. Billups, B.A.; *and others.*

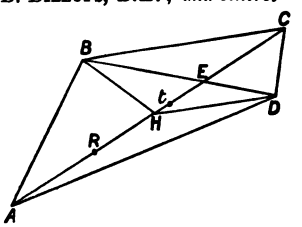
Bisect AC in H.

Then ts clearly divides HE, and therefore HB and HD in the ratio 1 : 2.

Therefore the centres of gravity of $\triangle$s ABC, ACD lie in ts.

Therefore the centre of gravity of ABCD lies in ts.

Similarly, it lies in t_1s, and is therefore at s.



13234. (Hugh MacColl, B.A.)—Prove the following formulæ in symbolic logic:—(1) $\alpha^x\beta^x = (\alpha\beta)^x$; (2) $\alpha^x\alpha^y = \alpha^{x+y}$; (3) $(\alpha)^y = \alpha^{xy}$;

(4) $\frac{x}{\alpha} \cdot \frac{y}{\alpha} = \frac{x+y}{\alpha}$; (5) $\frac{\alpha}{x} \cdot \frac{\alpha}{y} : \frac{\alpha}{xy}$; (6) $\frac{\alpha}{xy} = \alpha_{xy}\left(\frac{\alpha}{x} + \frac{\alpha}{y}\right)$.

Defs.—The symbol α^x is short for $\alpha + x'$, and asserts that either α is true or x false—possibly both. The symbol $\alpha\beta$ (or $\alpha.\beta$) asserts the truth of both α and β. The symbol $\alpha : \beta$ (or α_β) asserts that whenever α is true β is true. The *fractional* symbol α/β is short for $\alpha_\beta\,(\alpha'_\beta)'$; it asserts what α_β asserts, and *also* that β is *not* always true when α is *not* true.

Solution by H. W. Curjel, M.A.; Professor Lampe; *and others.*

The formulæ become obvious as soon as clearly interpreted. They may also be proved as follows:—

Let A, B, X denote the regions in which (or the conditions under which) α, β, x are true, and $\overline{A}$, $\overline{B}$, $\overline{X}$ the regions in which they are false; A + B the region in which one or both of α and β are true, and AB the region in which both are true, and so on.

Then we have $A + \overline{A} = 1$, $nA = A$, $A^n = A$,

$$(\overline{AB}) = \overline{A} + \overline{B}, \quad \overline{A}\,\overline{B} = (\overline{A + B}), \quad \text{and} \quad B + BA = B.$$

(1) $$(A + \overline{X})(B + \overline{X}) = AB + A\overline{X} + B\overline{X} + \overline{X}^2 = AB + \overline{X};$$

i.e., $$\alpha^x\beta^x = (\alpha\beta)^x.$$

(2) $$(A + \overline{X})(A + \overline{Y}) = A + \overline{X}\,\overline{Y} = A + (\overline{X + Y});$$

i.e., $$\alpha^x\alpha^y = \alpha^{x+y}.$$

(3) $$(A + \overline{X}) + \overline{Y} = A + \overline{X} + \overline{Y} = A + (\overline{X}\overline{Y});$$

$$\therefore \quad (\alpha^x)^y = \alpha^{xy}.$$

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Now $\alpha : \beta$ asserts $AB = .A$, which may also be written $A + B = B$; α/β asserts $AB = A$ and $\bar{A}B \neq \bar{A}$, *i.e.*, $B \neq 1$, or $A + B = B \neq 1$.

(4) Given $X + A = A \neq 1$ and $Y + A = A \neq 1$,

we have $X + Y + A = X + A = A \neq 1$.

Also given $X + Y + A = A \neq 1$,

we have $X + A = X + X + Y + A = X + Y + A = A \neq 1$,

$Y + A = X + Y + A + Y = X + Y + A = A \neq 1$;

i.e., $$\frac{x}{\alpha} \cdot \frac{y}{\alpha} = \frac{x+y}{\alpha}.$$

(5) Given $XA = A$, $YA = A$, we have $A(XY) = Y(XA) = YA = A$;

$$\therefore \frac{\alpha}{x} \cdot \frac{\alpha}{y} : \frac{\alpha}{xy}.$$

(6) Given $A + XY = XY \neq 1$,

we have $A + X = A + X + XY = X + XY = X$,

and $A + Y = A + XY + Y = XY + Y = Y$;

also $XY \neq 1$; therefore either $X \neq 1$, or $Y \neq 1$, or both.

And given $A + XY = XY$, and $A + X = X \neq 1$ or $A + Y = Y \neq 1$;

then $XY \neq 1$; $\therefore$ $A + XY = XY \neq 1$;

$$\therefore \frac{\alpha}{xy} = \alpha_{xy}\left(\frac{\alpha}{x} + \frac{\alpha}{y}\right).$$

[The PROPOSER sends the following solution:—In the following solutions the parenthetical symbols F_1, F_2, &c., refer to assumed formulæ which will be given, and (when necessary) proved afterwards. The symbol ϵ (as defined in 13288) denotes a *true* statement and its denial ϵ' a *false* statement. Hence, ϵ_a asserts that a is *always* true; and the denial of this, namely, ϵ_{0a} or $(\epsilon_a)'$, that a is *not* always true. The terms underlined are asserted to be *redundant* by virtue of the formula F_2.

(1) $\alpha^x\beta^x = (\alpha + x')(\beta + x') = \alpha\beta + \underline{\alpha x'} + \underline{\beta x'} + x' = (\alpha\beta)^x \ldots F_1F_2$.

(2) $\alpha^x\alpha^y = (\alpha + x')(\alpha + y') = \alpha + \underline{\alpha y'} + \underline{\alpha x'} + x'y' = \alpha^{x+y} \ldots F_1F_2F_3$.

(3) $(\alpha^x)^y = = \alpha^x + y' = \alpha + x' + y' = \alpha + (xy)' = \alpha^{xy} \ldots F_2$.

(4) $x/\alpha \,.\, y/\alpha = x_a\epsilon_{0a} \,.\, y_a\epsilon_{0a} = x_ay_a\epsilon_{0a} = (x+y)_a\,\epsilon_{0a} = (x+y)/\alpha \ldots F_3F_1F_4$.

(5) $\alpha/x \,.\, \alpha/y : \alpha_x\alpha_y\epsilon_{0x}\epsilon_{0y} : \alpha_x\alpha_y\epsilon_{0\,(xy)} : \alpha_{xy}\,\epsilon_{0\,(xy)} : \alpha/(xy)$.

(6) $\alpha/(xy) = \alpha_{xy}\,\epsilon_{0\,(xy)} = \alpha_{xy}\,(\epsilon_{0x} + \epsilon_{0y}) = \alpha_x\alpha_y\,(\epsilon_{0x} + \epsilon_{0y})$

$= \alpha_x\alpha_y\,(\alpha_x\epsilon_{0x} + \alpha_y\,\epsilon_{0y}) = \alpha_{xy}\,(\alpha/x + \alpha/y) \ldots F_3F_6F_7$.

The formulæ assumed in the preceding proofs are—

$(F_1)\ \alpha\alpha = \alpha = \alpha + \alpha$; $(F_2)\ \alpha\beta + \alpha = \alpha$; $F_3\,(\alpha\beta)' = \alpha' + \beta'$;

$(F_4)\ x_ay_a = (x+y)_a$; $F_5\,(x+y)_{0a} = x_{0a} + y_{0a}$; $F_6\,\alpha_{0\,(xy)} = \alpha_{0x} + \alpha_{0y}$;

$(F_7)\ \alpha\beta\,(\gamma + \delta) = \alpha\beta\,(\alpha\gamma + \beta\delta)$; $F_8\,\alpha_{,}\,\epsilon_{0\beta} = \alpha/\beta$.

Of these formulæ the only ones that do not seem quite axiomatic are F_4, F_5, F_6, F_8. We will first prove F_6.

$$\alpha_{0\,(xy)} = (\alpha_{xy})' = (\alpha_x\alpha_y)' = (\alpha_x)' + (\alpha_y)' = \alpha^0x + \alpha_{0y}.$$

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We will next prove F_8,

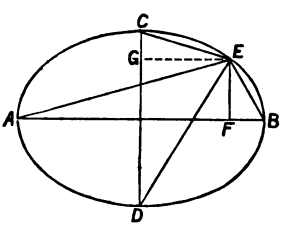
$$a_\beta \epsilon_{0\beta} = a_\beta (\alpha + \alpha')_{0\beta} = a_\beta (a_{0\beta} + a'_{0\beta}) = a_\beta a'_{0\beta} = a/\beta.$$

F_5 is an evident deduction from F_4, and F_4 becomes self-evident when illustrated geometrically as follows:—Let a circle ϵ wholly enclose a circle a, and let a wholly enclose two other circles x and y. Now take any point P at random in the outer circle ϵ, and let the letters ϵ, a, x, y assert respectively as propositions that P falls in ϵ, that P falls in a, that P falls in x, that P falls in y. Then the implication $x_a y_a$ may be read—"If P falls in x, it must fall in a; and, if it falls in y, it must fall in a"; while the equivalent implication $(x+y)_a$ may be read—"If P falls in x, or if P falls in y, then, in either case (separately or conjointly), it must fall in a."]

13233. (Editor.)—If θ, ϕ be the angles subtended by two conjugate diameters at a point in the curve of an ellipse, prove that $\cot^2\theta + \cot^2\phi$ is constant.

Solution by H. W. Curjel, M.A.; W. J. Greenstreet, M.A.; *and others.*

Let α, $\frac{1}{2}\pi+\alpha$, $\pi+\alpha$, $\frac{3}{2}\pi+\alpha$ be the excentric angles of the ends of the conjugate diameters. Let $a\cos\psi$, $b\sin\psi$ be any other point on ellipse.



Then we have $\cot\theta$

$$= \frac{1 + \dfrac{b^2\sin^2\alpha - b^2\sin^2\psi}{a^2\cos^2\alpha - a^2\cos^2\psi}}{\dfrac{b\sin\alpha + b\sin\psi}{a\cos\alpha + a\cos\psi} - \dfrac{b\sin\alpha - b\sin\psi}{a\cos\alpha - a\cos\psi}}$$

$$= \frac{(a^2-b^2)(\cos^2\alpha - \cos^2\psi)}{2ab\sin(\psi-\alpha)} = \frac{a^2-b^2}{2ab}\cos(\psi+\alpha).$$

Similarly, $$\cot\phi = \frac{(a^2-b^2)}{2ab}\cos(\psi+\alpha+\tfrac{1}{2}\pi);$$

hence we have $$\cot^2\theta + \cot^2\phi = \frac{(a^2-b^2)^2}{4a^2b^2}.$$

13245. (H. J. Woodall, A.R.C.S.)—Prove that $2^{155}-1$ is divisible by $(31)^2$.

Solution by the Proposer; Rev. J. L. Kitchin, M.A.; *and others.*

In Question 13087 put $a=2$, $x=5$, $y=31$, and the result follows immediately. It can be easily proved that 155 is the least value of x for which 2 (Exp. x) -1 is divisible by $(31)^2$.

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13247. (R. F. Davis, M.A.) — Prove that the cotangent of the Brocard-angle (1) of any triangle is $\frac{1}{2}\{s/r-(4R+r)/s\}$, (2) of the triangle formed by joining the points of contact of the sides of the original triangle with the inscribed circle is $(4R+r)/s$.

Solution by H. W. Curjel, M.A.; W. E. Jeffares, B.A.; *and others.*

$$\operatorname{Cot}\omega = \frac{a^2+b^2+c^2}{4S} = \tfrac{1}{2}\frac{2s^2-(ab+bc+ca)}{S} = \tfrac{1}{2}\left\{\frac{s}{r}-\frac{ab+bc+ca-s^2}{4Ss}\right\}$$

$$= \tfrac{1}{2}\left\{\frac{s}{r}-\frac{abc+(s-a)(s-b)(s-c)}{Ss}\right\} = \tfrac{1}{2}\left\{\frac{s}{r}-\frac{4R+r}{s}\right\}.$$

Cot of Brocard-angle of contact Δ of in-circle

$$= \Sigma\cot\tfrac{1}{2}(B+C) = \Sigma\tan\tfrac{1}{2}A = \frac{ab+bc+ca-s^2}{S} = \frac{4R+r}{s}.$$

13207. (R. F. Davis, M.A.)—Prove that (1) the sides of a triangle are the roots of the equation $x^3-2sx^2+(s^2+r^2+4Rr)x-4Rrs=0$; and (2) the cosines of the angles are the roots of

$$4R^2x^3-4R(R+r)x^2+x(s^2+r^2-4R^2)-s^2+(2R+r)^2=0,$$

so that the triangle is acute-angled or obtuse-angled according as s is greater or less than $2R+r$.

Solution by W. J. Greenstreet, M.A.; H. B. Billups, B.A.; *and others.*

$x^3-fx^2+gx-h=0$ has roots $(s-a)$, $(s-b)$, $(s-c)$, where

$$f=\Sigma(s-a)=s,\quad g=\Sigma(s-a)(s-b)=\Sigma(\Delta^2r_3/r_1r_2r_3)=r(r+4R),$$
$$h=\Pi(s-a)=\Delta^2/s=r^2s;$$

therefore $x^3-sx^2+r(r+4R)x-r^2s=0$ has roots $(s-a)$, $(s-b)$, $(s-c)$.

Write $x=s-y$. Then, when $x=s-a$, $y=a$, therefore a, b, c are roots of $y^3-2sy^2+(r^2+s^2+4Rr)y-4Rrs=0$.

Again, $\Sigma\cos A = 1+4\Pi\sin\frac{1}{2}A = 1+(r/R) = (R+r)/R$.

Also $\Sigma\sin A = \Sigma(a/2R) = s/R$;

therefore, squaring and subtracting,

$$\Sigma\cos 2A+2\Sigma\cos(B+C) = \{1+(r/R)\}^2-(s^2/R^2) = -1-4\Pi\cos A-2\Sigma\cos A;$$
$$\therefore\quad \Pi\cos A = \tfrac{1}{4}(s^2/R^2)-\tfrac{1}{4}\{2+(r/R)\}^2.$$
$$\Sigma\cos B\cos C = \tfrac{1}{2}(\Sigma\cos A)^2-\tfrac{1}{2}\Sigma\cos^2 A = \tfrac{1}{2}\{(R+r)/R\}^2-\tfrac{3}{4}-\tfrac{1}{4}\Sigma\cos 2A$$
$$= (s^2+r^2-4R^2)/4R^2;$$
$$\therefore\quad 4R^2x^3-4R(R+r)x^2+x(s^2+r^2-4R^2)-\{s^2-(2R+r)^2\}=0$$

has cos A, &c., for its roots; and, if $s>(2R+r)$, the cosines are all positive, *i.e.*, Δ is acute-angled; if $s<(2R+r)$, one of the three is negative, *i.e.*, Δ is obtuse-angled.

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11652. (Professor MORLEY.)—Denoting the absolute value of $x+iy$, *i.e.*, $+(x^2+y^2)^{\frac{1}{2}}$, by $|\,x+iy\,|$, prove

$$|\,z+(z^2-c^2)^{\frac{1}{2}}\,|\,+\,|\,z-(z^2-c^2)^{\frac{1}{2}}\,|\,=\,|\,z+c\,|\,+\,|\,z-c\,|\,.$$

Solution by H. W. CURJEL, M.A.; Prof. RAMASWAMI AIYAR; *and others.*

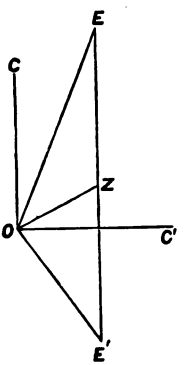
Let O be the origin, and let z and c be represented by points Z, C.

Draw OC′ at right angles to and = OC.

Then, if we draw EZE′ parallel to OC, making

$$E'Z = ZE = OC,$$

then OE, OE′ will be equal to the sum and difference of the axes of the ellipse which has OZ, OC′ as conjugate semi-diameters, and the foci are given by $\pm(z^2-c^2)^{\frac{1}{2}}$;



$\therefore\ |\,z+c\,|\,+\,|\,z-c\,|$

$= OE + OE' =$ major axis

= sum of focal distances of Z

$= |\,z+(z^2-c^2)^{\frac{1}{2}}\,|\,+\,|\,z-(z^2-c^2)^{\frac{1}{2}}\,|\,.$

4536. (Rev. W. ROBERTS, M.A.)—Adopting the ordinary notation of the theory of elliptic functions, prove that

$$\int_0^{\frac{1}{2}\omega}\int_0^{\frac{1}{2}\omega}\log\left\{1-k^2\sin^2 am\left(\frac{2\mathrm{K}x}{\pi}\right)\sin^2 am\left(\frac{2\mathrm{K}y}{\pi}\right)\right\}dx\,dy$$

$$=\frac{\pi^2}{12}\left\{\log\left(\frac{4k'^2}{k}\right)-\frac{\pi}{2}\frac{\mathrm{K}'}{\mathrm{K}}\right\}.$$

Solution by Rev. J. CULLEN, B.A.; Professor COONDOO; *and others.*

Let $2\mathrm{K}x/\pi = u$ and $2\mathrm{K}y/\pi = v$. Then we have

$$\mathrm{I}=\frac{\pi^2}{4\mathrm{K}^2}\int_0^{\mathrm{K}}\int_0^{\mathrm{K}}\log(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)\,du\,dv.$$

Now $\Theta(u+v)\,\Theta(u-v)=\frac{\pi}{2k'\mathrm{K}}\,\Theta^2 u\,\Theta^2 v\,(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)$

(CAYLEY'S *Elliptic Functions*, Art. 207);

$$\therefore\ \mathrm{I}=\frac{\pi^2}{4\mathrm{K}^2}\int_0^{\mathrm{K}}\int_0^{\mathrm{K}}\{\log\Theta(u+v)+\log\Theta(u-v)-\log(\pi/2k'\mathrm{K})$$
$$-2\log\Theta u-2\log\Theta v\}\,du\,dv \quad \ldots\ (1).$$

Also $\Theta(z)=(2k'\mathrm{K}/\pi)^{\frac{1}{2}}\left[1-2q^{2m-1}\cos(\pi z/\mathrm{K})+q^{4m-2}\right]/\left[1-q^{2m-1}\right]^2$ ..... (2).

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Whence, employing (2) in (1), and expanding

$$\log\left[1-2q^{2m-}\ \cos(\pi/\mathrm{K})\,(u+v)+q^{4m-2}\right],\ \text{&c.},$$

we have $\mathrm{I}=\frac{\pi^2}{2\mathrm{K}^2}\int_0^{\mathrm{K}}\int_0^{\mathrm{K}}\log\left[1-q^{2m-1}\right]^2 du\,dv=\frac{1}{2}\pi^2\log\left[1-q^{2m-1}\right]^2$

$$=\tfrac{1}{8}\pi^2\log\left[1-q^{2m-1}\right]^6.$$

But $\left[1-q^{2m-1}\right]^6=2\sqrt[3]{q}\,k'/\sqrt{k}$ (*ib.* Art. 376);

$$\therefore\ \mathrm{I}=\frac{\pi^2}{12}\left\{\log\left(\frac{4k'^2}{k}\right)-\frac{\pi}{2}\frac{\mathrm{K}'}{\mathrm{K}}\right\},\ \text{since}\ q=e^{-\pi\mathrm{K}'/\mathrm{K}}.$$

4364. (Rev. G. H. Hopkins, M.A.)—In what year before Christ did the sun enter each sign of the zodiac upon the first day of the month to which such sign corresponded?

Solution by Professors Sanjána, Ramaswami Aiyar, *and others.*

The difficulty in the problem arises from our not knowing precisely the initial point of each sign as determined by the ancients; possibly they had no such points, and in the infancy of observation the signs probably did not divide the Ecliptic into 12 mathematically equal parts.

The Vernal Equinox moves from east to west among the stars; hence the First Point in the constellation of Aries, when so named, probably coincided in longitude with the most noticeable of its eastern stars. Such a star is δ Arietis. On January 1, 1891, its R.A. was 3 h. 5 m. 23·72 s., and its d 19° 18′ 50·63″; also ω was 23° 27′ 12·31″ on that day. Hence, by the usual formulæ,

$$\cot\phi=\sin\text{R.A.}\cot d,\quad \tan l=\cos(\phi-\omega)\tan\text{R.A.}/\cos\phi,$$

we obtain the value 49° 19′ 35·62″ for l, its longitude on that day. This value, therefore, gives the total amount of precession up to the epoch. At the rate of 50·2″ per year for precession, the era of observation comes out as 3549 − 1890 = 1659 B.C. If we take the more accurate value of 25700 years for the period of precession—for the rate was slightly larger in former times—the date comes out as 3521 − 1890 = 1631 B.C. [There is a similar difficulty in determining the date of the *Brahmanas*, from the positions of the Vernal Equinox in the Lunar Asterisms, as recorded in ancient Hindu books.]

7206. (D. Edwardes.)—Prove that

$$\int_0^{\mathrm{K}}\frac{(\mathrm{cn}^2u+k'^4\,\mathrm{sn}^2u)^{\frac{1}{2}}}{\mathrm{dn}^2u}du=\frac{1}{k'}\,\mathrm{E}^1(k),$$

where $k'^2=1-k^2$, and $\mathrm{E}^1(k)$ denotes the complete elliptic integral of the second kind whose modulus is k.

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Solution by Rev. J. CULLEN, B.A.; Prof. MUKHOPADHYAY; *and others.*

Since $\operatorname{sn}(K-u) = \operatorname{cn} u/\operatorname{dn} u$, $\operatorname{cn}(K-u) = k' \operatorname{sn} u/\operatorname{dn} u$,

and as the integral may be written thus:

$$I = \int_0^K (\operatorname{cn}^2 u/\operatorname{dn}^2 u + k'^4 \operatorname{sn}^2 u/\operatorname{dn}^2 u)^{\frac{1}{2}}\, du/\operatorname{dn} u,$$

we have $I = \int_0^K \left[\operatorname{sn}^2(K-u) + k'^2 \operatorname{cn}^2(K-u)\right]^{\frac{1}{2}} du/\operatorname{dn} u$

$$= \int_0^K \left[1 - k^2 \operatorname{cn}^2(K-u)\right]^{\frac{1}{2}} du/\operatorname{dn} u \qquad (k'^2 = 1-k^2)$$

$$= \int_0^K (1 - k^2 \operatorname{cn}^2 v)^{\frac{1}{2}}\, dv/\operatorname{dn}(K-v) \qquad (v = k-u)$$

$$= \frac{1}{k'} \int_0^K (1 - k^2 \operatorname{cn}^2 v)^{\frac{1}{2}} \operatorname{dn} v\, dv$$

$$\{\operatorname{dn}(K-v) = \left[1 - k^2 \operatorname{sn}^2(K-v)\right]^{\frac{1}{2}} = k'/\operatorname{dn} v\}$$

$$= \frac{1}{k'} \int_0^{\frac{1}{2}\pi} (1 - k^2 \cos\phi)^{\frac{1}{2}}\, d\phi \qquad (\sin\phi = \operatorname{sn} v;\ \therefore\ d\phi = \operatorname{dn} v\, dv)$$

$$= \mathrm{E}_1(k)/k', \text{ by writing } \phi = \tfrac{1}{2}\pi - \theta.$$

[For another solution, see Vol. LIX., p. 103.]

4625. (The late Rev. J. BLISSARD.)—Prove that

(1) $\ldots \left\{\sin x \cdot \frac{\Delta}{1} - \frac{\sin 2x}{2\cos x} \cdot \frac{\Delta^2}{2} + \frac{\sin 3x}{(2\cos x)^2} \cdot \frac{\Delta^3}{3} - \&c.\right\} 0^n = 0 \quad (n \text{ even}),$

(2) $\ldots \left\{\cos x \cdot \frac{\Delta}{1} - \frac{\cos 2x}{2\cos x} \cdot \frac{\Delta^2}{2} + \frac{\cos 3x}{(2\cos x)^2} \cdot \frac{\Delta^3}{3} - \&c.\right\} 0^n = 0 \quad (n \text{ odd and } > 1).$

Solution by Rev. J. CULLEN, B.A.; Professor CHAKRIVARTI; *and others.*

By HERSCHEL'S Theorem, we have

$$\phi(e^t - 1) = \phi(\Delta)\, 0 \,.\, t + \phi(\Delta)\, 0^2 \,.\, t^2/2! + \ldots + \phi(\Delta)\, 0^n \,.\, t^n/n! + \ldots.$$

Putting $\Sigma(-)^{n+1} \sin nx\, \Delta^n/n\,(2\cos x)^n$ for $\phi(\Delta)$, we see that the coefficient of $t^n/n!\, 2\cos x$ is the expression on the left-hand side of (1)

Now $\phi(e^t - 1)$, or, say, $\psi(t) = \Sigma(-)^{n+1} \sin nx\, (e^t-1)^n\, n\,(2\cos x)^n$

$$= \cot^{-1}\left\{\frac{e^t+1}{e^t-1}\cot x\right\};$$

$$\therefore \quad \psi(-t) = \cot^{-1}\left\{-\frac{e^t+1}{e^t-1}\cot x\right\} = -\psi(t).$$

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Hence $\psi(t)$ is an odd function of t, *i.e.*, the coefficient of $t^n = 0$, when n is even, which gives the first result in the question.

Similarly, in the second case, writing $\Sigma(-)^{n+1}\cos nx\,\Delta^n/n\,(2\cos x)^n$ for $\phi(\Delta)$ and $\psi(t)$, as before, we have

$$\psi(t) = \tfrac{1}{2}\log\left\{1+(4\cos^2 x-2)\,e^t+e^{2t}\right\}-\log(2\cos x);$$

$$\therefore\quad \psi(-t) = \psi(t)-t,$$

which shows that $\psi(t)$ does not contain an odd power of t after the first. Therefore the coefficient of $t^n = 0$, if n is odd and > 1.

13228. (Professor Barisien.)—L'équation du 4e degré

$$(b^2z^2-a^2)^2-4ab\,(bz^2-a)(az-b) = 0$$

est décomposable en un produit de deux facteurs du second degré. Des racines de cette équation, deux sont toujours réelles; les deux autres, toujours imaginaires.

Solution by H. J. Woodall, A.R.C.S.; W. J. Greenstreet, M.A.; *and others.*

The equation when expanded can be easily decomposed into

$$\left\{z^2(b^2-2ab)+2abz+(a^2-2ab)\right\}\times\left\{z^2(b^2+2ab)+2abz+(a^2+2ab)\right\} = 0.$$

The roots of the first factor are real if

$$-2ab\,(a-b)^2 < 0;$$

but this is the condition that the roots of the second factor are imaginary.

13240. (R. Chartres.)—If the sides of a triangle be in arithmetical progression, and the bisector of the greatest angle $(1/m)^{\text{th}}$ of the perimeter, find in lowest integers (1) the sides and bisector, and (2) the sides, bisector, and m; and (3) the greatest angle in each case.

Solution by D. Biddle, *the* Proposer, *and others.*

Let $x-y$, x, $x+y$ be the three sides; then the bisector of the greatest angle $= 3x/m = k$. Let u, v be the segments of the base; then

$$u = (x^2-y^2)/(2x-y),\qquad v = x(x+y)/(2x-y),$$

and we have
$$\frac{k^2+u^2-(x-y)^2}{2ku}+\frac{k^2+v^2-x^2}{2kv} = 0,$$

or
$$(2m^2-3)\,y^2-(3m^2-12)\,x.y+(m^2-12)\,x^2 = 0;$$

whence
$$\frac{y}{x} = \frac{3(m^2-4)\pm m\,(m^2+36)^{\frac{1}{2}}}{2(2m^2-3)}.$$

Let $36 = q(2m+q)$; then $(m^2+36)^{\frac{1}{2}} = m+q$.

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By a series of trials, we obtain—

(1) $q = 3$, whence $m = 4\frac{1}{2}$ and $y/x = \frac{1}{2}$. The lowest integers are 12, 15, 18 for the sides, and 10 for the bisector; also 8 and 10 for the segments.

(2) $q = 2$, whence $m = 8$ and $y/x = \frac{2}{3}$. The lowest integers are 24, 40, 56 for the sides, 15 for the bisector, and 8, as we have seen, for m; also 21 and 35 for the segments of the base.

(3) In (1), the cosine of the greatest angle = ·125000 = cos 82° 49′ 9¼″. In (2), the cosine = −·500000 = cos 120°.

13250. (Professor Neuberg.) — Les carrés des côtés d'un triangle ABC sont racines de l'équations $A_0x^3 + 3A_1x^2 + 3A_2x + A_3 = 0$. Trouver l'angle de Brocard de ABC.

Solution by Rev. J. Cullen, B.A.; W. E. Jeffares, B.A.; *and others.*

We have $$\omega = \cot^{-1}\{\Sigma a^2/4\Delta\},$$

and $$\Delta = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}} = [4\Sigma a^2b^2 - (\Sigma a^2)^2]^{\frac{1}{2}}/4;$$

also $$\Sigma a^2 = -3A_1/A_0 \quad \text{and} \quad \Sigma a^2b^2 = 3A_2/A_0;$$

$$\therefore \quad \omega = \cot^{-1}\{-3A_1/[12A_0A_2 - 9A_1^2]^{\frac{1}{2}}\}.$$

13211. (R. Knowles, B.A.)—Rectangular hyperbolas are drawn to have contact of the second order with the parabola $y^2 = 4ax$, at a point P $(am^2, 2am)$; prove that the locus of the mid-points of their common chords is the parabola $y^2 - 2ax - 2amy + 2a^2m^2 = 0$.

Solution by Professor Sanjána; H. B. Billups, B.A.; *and others.*

A rectangular hyperbola is determined by four points; in the present case, three of the points are at P. Take a fourth point anywhere on the parabola; then there will be only one hyperbola passing through this point and having contact at P of the second order. Thus the required locus is that of the middle points of chords of the given parabola passing through P. This, with ordinary working, is found to be the parabola $$y^2 = 2a(x + my - am^2).$$

1451. (Dr. Hirst, F.R.S.)—Prove that, if

$$F(x, y, a, b, c, f, g, h) = 0$$

be the equation of the n^{th} pedal of the conic

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0,$$

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the equation of the $(-n+1)^{\text{th}}$ pedal will be

$$F\left(-\frac{x}{r^2},\ -\frac{y}{r^2},\ \frac{d\Delta}{da},\ \frac{d\Delta}{db},\ \frac{d\Delta}{dc},\ \tfrac{1}{2}\frac{d\Delta}{df},\ \tfrac{1}{2}\frac{d\Delta}{dg},\ \tfrac{1}{2}\frac{d\Delta}{dh}\right)=0,$$

where $r^2=x^2+y^2$, and $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution by Rev. J. CULLEN, B.A.; Professor COLAW; *and others.*

If, as in the solution to Quest. 1442 (Vol. I., p. 42), we write J for the operation of inversion, R for that of reciprocation, and P for that of taking the pedal, the same circle (radius $= k$) around the origin being employed in inversion and reciprocation,

we have $J^2=R^2=1$, $JR=P$, $RJ=P^{-1}$;

$$\therefore\ JP^nR=J(JR)^nR=J.JR(JR)^{n-1}R=R(JR)^{n-1}J.JR$$
$$=(RJ)^nJR=P^{-n+1};$$

which may thus be interpreted: "The inverse of the n^{th} pedal of the reciprocal is the $(-n+1)^{\text{th}}$ pedal of the primitive."

Now the reciprocal of

$$(a, b, c, f, g, h)(x, y, 1)^2=0 \quad \ldots\ldots\ldots\ldots\ (1)$$

is $$\left(\frac{d\Delta}{da},\ \frac{d\Delta}{db},\ \frac{d\Delta}{dc},\ \tfrac{1}{2}\frac{d\Delta}{df},\ \tfrac{1}{2}\frac{d\Delta}{dg},\ \tfrac{1}{2}\frac{d\Delta}{dh}\right)\left(-\frac{x}{k^2},\ -\frac{y}{k^2},\ 1\right)^2=0 \ \ldots\ (2)$$

(SALMON, *Conics*, p. 278), and the n^{th} pedal of (2) is

$$F\left(-\frac{x}{k^2},\ -\frac{y}{k^2},\ \frac{d\Delta}{da},\ \frac{d\Delta}{db},\ \frac{d\Delta}{dc},\ \tfrac{1}{2}\frac{d\Delta}{df},\ \tfrac{1}{2}\frac{d\Delta}{dg},\ \tfrac{1}{2}\frac{d\Delta}{dh}\right)=0 \ldots\ldots\ (3).$$

The inverse of (3) or the $(-n+1)^{\text{th}}$ pedal of (1) is obtained by writing xk^2/r^2 and yk^2/r^2 for x and y in (3), which gives the result in the question.

13238. (Rev. J. CULLEN, B.A.)—Solve the difference equation

$$u_{x+1}u_x+x^2(u_{x+1}+u_x)+x^4+4x^2-1=0.$$

Solution by H. W. CURJEL, M.A.; Professor COONDOO; *and others.*

Let $$u_x=\frac{(2x-1)w_{x+1}}{w_x}-x^2.$$

Then $w_{x+2}-w_{x+1}+w_x=0$; $\therefore\ w_x=C\sin\left\{\frac{1}{3}(\pi x+\alpha)\right\}$;

$$\therefore\ u_x=\frac{(2x-1)\sin\frac{1}{3}\left\{\pi(x+1)+\alpha\right\}}{\sin\frac{1}{3}(\pi x+\alpha)}-x^2$$
$$=-x^2+x-\tfrac{1}{2}+(2x-1)\tfrac{1}{2}\sqrt{3}\cot\tfrac{1}{3}(\pi x+\alpha).$$

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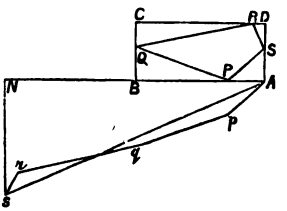
4682. (Professor Evans.)—Find the quadrilateral of minimum perimeter that can be inscribed in a given rectangle so that one of its sides shall pass through a given point within the rectangle.

Solution by H. W. Curjel, M.A.; Prof. Radhakrishnan; *and others.*

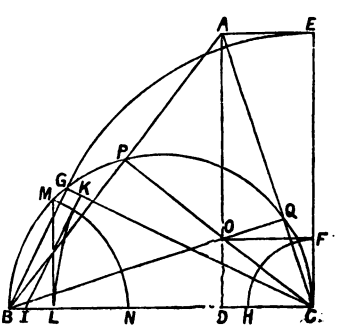


Let PQRS be a quadrilateral inscribed in the rectangle ABCD. Draw Ap equal and parallel to SP, and pq equal and parallel to the reflexion of PQ with respect to AB; and qr equal and parallel to RQ, and rs equal and parallel to the reflexion of SR with respect to AB. Draw sN perpendicular to AB. Then AN clearly = 2AB and Ns = 2CB; therefore s is a fixed point, and As = 2AC and is parallel to BD. Therefore the perimeter of PQRS is least when A$pqrs$ is a straight line, *i.e.*, when the sides of PQRS are parallel to the diagonals of ABCD, and in that case the perimeter = 2AC.

Hence construction: Through given point E draw a parallel to either diagonal, meeting the sides in F and G, and through F and G draw parallels to the other diagonal, meeting the sides in K and H. Then FGHK is the required quadrilateral, for it is a parallelogram with its sides parallel to the diagonals of the rectangle. There will be two such parallelograms with a side passing through E.

13279. (Professor Sanjána, M.A.)—Given the two segments of an altitude made by the orthocentre and the corresponding base of a triangle; construct it.

Solution by D. Biddle; Prof. Mukhopadhyay; *and others.*



Let BC be the given side, and CF, FE the given segments. On BC describe the semi-circle BPQC, and from C as centre draw the arcs EG, FH. Join CG, GB, and, having made CI = 4CH, draw IK parallel to BG. Make CL = CK, and draw LM at right angles to BC. Make BN = BM, and bisect NC in D. Draw DA at right angles to BC, and draw EA parallel to BC (CE being perpendicular to it). AB, AC being joined, ABC is the required triangle, and CP, BQ intersect AD in O, defined by

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FO parallel to BC. For let CD = x, whilst CF = k, and FE = l. Then $k/x = \cot B$, $k/(a-x) = \cot C$, $(k+l)/(a-x) = \tan B$, $(k+l)/x = \tan C$.

$\therefore\ k(k+l)/(ax-x^2) = 1$, whence $x = \frac{1}{2}\left[a \pm \{a^2 - 4k(k+l)\}^{\frac{1}{2}}\right]$.

This formula is faithfully carried out in the construction.

4665. (Professor CROFTON, F.R.S.)—Four beams are jointed together, forming a parallelogram, the frame being kept in equilibrium by any four given forces at the joints, distortion being prevented by the stiffness of the joints. Prove that, if the bending moment on each joint be calculated as if it alone were to resist the distortion, the values will be the same for all four joints. What will the value be? For *any* four-sided frame, at which joint would the greatest bending stress occur, if the remaining three joints are supposed free, as before?

Solution by H. W. CUILJEL, M.A.; Professor LAMPE; *and others.*

Let ABCD be the quadrilateral, and α, β, γ, δ the bending moments. Resolve the forces at each angular point along the sides which meet in that point. This reduces the system to P, Q, R, S along AB, BC, CD, DA; and the resultant of P and Q and that of R and S are equal and opposite acting along BD, and the resultant of Q and R and that of S and P are equal and opposite acting along AC.

Fig 2.

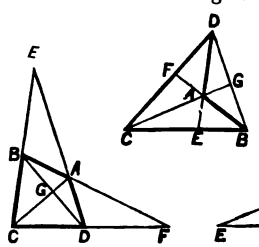

Fig. 1.

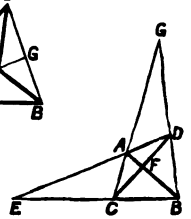

Fig. 3.

Fix the points A and B and take the moment about A of the force which does not pass through a fixed point; this gives α. Hence, if ABCD is a parallelogram, $\alpha = \beta = \gamma = \delta = P \times$ distance between AB and CD. If ABCD is not a parallelogram, let CB and DA meet in E, BA and CD in F, AC and BD in G; and let A be within $\triangle EFG$. Then α will be the least of α, β, γ, δ.

Since A is nearer than D to BC, and nearer than B to CD, $\alpha < \delta$ and $\alpha < \beta$. Also, since the parallelogram with diagonal AC and sides along CB, CD is less than that with diagonal AC and sides along AB, AD, $\alpha < \gamma$; therefore α is the least of α, β, γ, δ.

In Figs. 1 and 2 the forces at an angular point act both towards or both from it; in Fig. 3 the forces at an angular point act one towards and the other from it.

If AB is parallel to CD and < CD, $\alpha = \beta < \gamma = \delta$.

If AC is parallel to and < BD, $\alpha = \gamma < \beta = \delta$.

If AD is parallel to BC and AC to BD, $\alpha = \beta = \gamma = \delta$.

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4673. (S. WATSON.)—Through the angles of a given triangle straight lines are drawn at random, but so that a portion of each falls within the triangle. Find the average area of the triangle formed by their intersections.

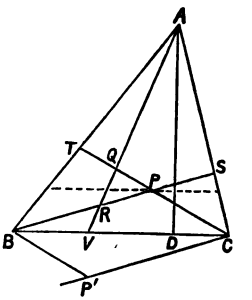
Solution by D. BIDDLE; Professor SANJÀNA; *and others.*

Let BS, CT, AV be the three lines, and PQR the enclosed triangle. Then

$$PQR = ABC - (PBC + QCA + RAB),$$

and the average PQR

$$= ABC - (\text{aver. } PBC + \text{aver. } QCA + \text{aver. } RAB).$$



If we attempt to find these averages by ordinary methods, we speedily become involved in an attempt to integrate compound logarithmic terms which refuse to yield. The following tentative method is therefore submitted:—

Suppose n lines drawn across the triangle from B; then the number supposed to be drawn from C must be also n, and not proportionate to the angle. Similarly with A. The infinitesimal angles separating two adjacent lines in each series are therefore as

$$3B/(A+B+C),\ 3C/(A+B+C),\ 3A/(A+B+C),\ \equiv k : l : m.$$

In an equilateral triangle, these values are equal and may be disregarded.

Taking PBC first, it will be observed that the interspaces between P and the nearest intersections of the series of lines drawn from B and C are parallelograms whose angles are BPC and $\pi - BPC$, like those of BPCP'. But the sides of the lozenge-shaped interspaces are as $k.BP : l.PC$, and these are respectively measured along PC, BP. Accordingly the density of the apices represented by P varies inversely as $k.l.\Delta$, where Δ is the triangle formed by the variable apex with the base BC. Let AD = unity, and the height of $\Delta = x$. Then the density of apices is the same at all parts of an intercept parallel to BC, being given by $1/(k.l.a.x)$. The intercept varies as $1-x$, and the average height of the elemental spaces is that of the centre of gravity of ABC. Therefore, multiplying by x, we obtain

$$\frac{1}{k.l.a} = \frac{\text{average height of apices}}{\text{average height of elements}},$$

and $AD/(3k.l.a)$ is the mean PBC/ABC. Consequently, by substitution and transposition of values, we have

$$\frac{\text{average PQR}}{ABC} = 1 - \frac{\pi^2}{27}\left(\frac{c \sin B}{B.C.a} + \frac{a \sin C}{C.A.b} + \frac{b \sin A}{A.B.c}\right).$$

In the equilateral triangle, the proportionate area $PQR/ABC = 1 - \frac{1}{2}\sqrt{3}$.

N.B.—The above result for PQR/ABC might be given wholly in terms of the angles, thus—

$$1 - \frac{\pi^2}{27}\left\{\frac{1}{B.C(\cot B + \cot C)} + \frac{1}{C.A(\cot C + \cot A)} + \frac{1}{A.B(\cot A + \cot B)}\right\}.$$

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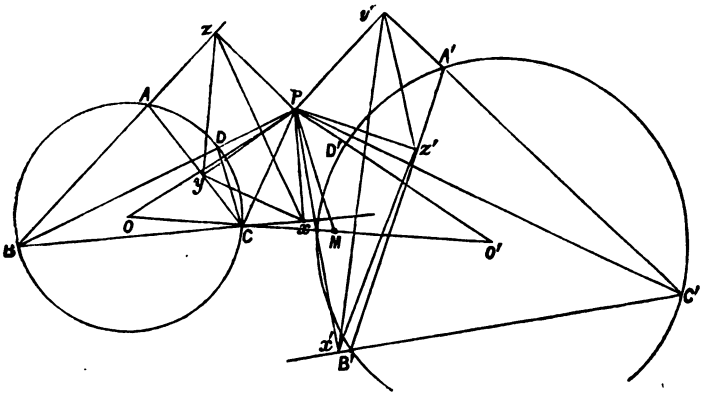
13278. (Professor Neuberg.)—Trouver le lieu d'un point tel que les aires de ses triangles podaires par rapport à deux triangles donnés ABC, A′B′C′ aient une somme donnée.

Solution by W. E. Jeffares, B.A.; Prof. Ramaswami Aiyar; *and others.*

Let ABC and A′B′C′ be the two triangles with their circumcircles, O and O′ their centres, R and R′ their radii, and P the point of which we require the locus.



Let fall Px, Py, Pz perpendiculars on the sides of ABC, and Px', Py', Pz' on those of A′B′C′. Then xyz and $x'y'z'$ are the pedal triangles of ABC and A′B′C′ with respect to the point P. Now PxBz is a cyclic quadrilateral, and PB is the diameter of its circumcircle; therefore $zx = \text{PB} \sin \text{B}$; similarly, $xy = \text{PC} \sin \text{C}$, since P$xCy$ is a cyclic quadrilateral, and PC the diameter of its circumcircle. The area of the pedal triangle

$$yzx = \tfrac{1}{2}zx \times xy \sin zxy = \tfrac{1}{2}\text{PB} \cdot \text{PC} \sin \text{B} \sin \text{C} \sin zxy,$$

$$\angle \text{PCD} = \text{PC}y - \text{DCA} = \text{P}xy - \text{PBA} = \text{P}xy - \text{P}xz = zxy;$$

therefore $$\angle \text{PCD} = zxy;$$

therefore $$2\Delta zxy = \text{PB} \cdot \text{PC} \sin \text{B} \sin \text{C} \sin \text{PCD};$$

but $$\frac{\text{PC}}{\text{PD}} = \frac{\sin \text{A}}{\sin \text{PCD}}; \quad \text{therefore} \quad \text{PC} = \text{PD} \frac{\sin \text{A}}{\sin \text{PCD}};$$

therefore $$2\Delta zxy = \text{PB} \cdot \text{PD} \sin \text{A} \sin \text{B} \sin \text{C}.$$

Also, similarly, $2\Delta z'x'y' = \text{PC}' \times \text{PD}' \sin \text{A}' \sin \text{B}' \sin \text{C}'$.

Take $m = \sin \text{A} \sin \text{B} \sin \text{C}$ and $n = \sin \text{A}' \sin \text{B}' \sin \text{C}'$;

we have $$2(\Delta zxy + \Delta z'x'y') = m\text{PB} \cdot \text{PC} + n\text{PC}' \times \text{PD}'$$
$$= m(\text{PO}^2 - \text{R}^2) + n(\text{PO}'^2 - \text{R}'^2);$$

therefore $$2(\Delta zxy + \Delta z'x'y') + m\text{R}^2 + n\text{R}'^2 = m\text{PO}^2 + n\text{PO}'^2.$$

Now, since the sum of the areas of the pedal triangles is given, and $m\text{R}^2 + n\text{R}'^2$ is given, $m\text{PO}^2 + n\text{PO}'^2$ is given. Now it reduces to a well-

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known problem : Given the base of a triangle and m squares of one side $+n$ squares of the other, to find the locus (of vertex). The locus is a circle with the centre at the point M which divides the base OO′ internally. So that OM : O′M = n : m and radius

$$= mPO^2 + nPO'^2 - (mOM^2 + nO'M^2) \text{ divided by } (m+n).$$

If the sum of areas was a minimum, the point P would be at M.

(2) It would be also a circle if the difference of areas was given. Its centre would be the point dividing OO′ externally in the ratio of n : m, and its radius equal $mPO^2 - nPO'^2 - (mBM'^2 - nCM'^2)$ divided by $m-n$, where M is the point of external section ; when P is at M′ the difference of areas will be a minimum.

(3) It would be a circle if the ratio of the areas was given. In this case the circle would be coaxal with the two circumcircles. The solution of this would be the same as the solution sent a short time ago for Question 4259 by Elizabeth Blackwood.

13153. (Rev. T. W. Robinson, B.N.) — POP′ and QOQ′ are two straight lines inclined to one another at an angle C; OP and OQ are equal to $\tan \frac{1}{2}b$ and $\tan \frac{1}{2}a$, and OP′ and OQ′ to $\cot \frac{1}{2}b$ and $\cot \frac{1}{2}a$, where a, b, and C are two sides and included angle of a spherical triangle ABC. Prove the angles PQ′P′ and Q′P′Q are equal to the angles B and A of the spherical triangle, and the angle PQ′O is equal to half the spherical excess.

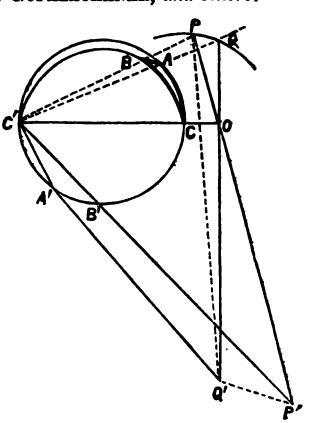
Solution by the Proposer, Professor Gopalachanar, *and others.*

If CAB be spherical triangle, and C′A′B′ triangle diametrically opposite to CAB, then, if we stereographically project from C′ on to plane parallel to tangent plane at C,

$$OQ = \tan \tfrac{1}{2}b, \quad OQ' = \cot \tfrac{1}{2}b,$$

$$OP = \tan \tfrac{1}{2}a, \quad OP' = \cot \tfrac{1}{2}a,$$

$$\angle POQ = C.$$



Now PQ, P′Q′ lie on a circle—projection of circle ABA′B′;

$\therefore$ $\angle P'Q'P$ = angle between OP and arc PQ

= angle between CA and BA (since angles project into equal angles stereographically)

$= \angle CAB.$

Similarly $\angle Q'P'Q = \angle CBA$; but $\angle C = \angle Q'OP'$;

$\therefore$ evidently $\angle PQ'O = \frac{1}{2}$ spherical excess.

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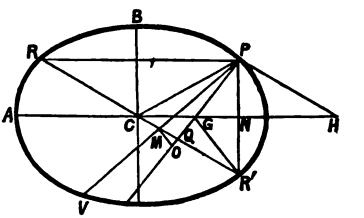
13300. (R. KNOWLES, B.A.)—The circle of curvature centre O is drawn at a point P of a conic, M is the mid-point of their common chord; the diameter of the conic through M meets the normal at P in Q; prove (1) that Q is the point through which pass all chords of the conic that subtend a right angle at P; (2) the ratio OQ : OM is constant.

Solution by W. C. STANHAM, M.A.; Rev. J. CULLEN, B.A.; *and others.*



PV is the chord of curvature, PG the normal, PR and PR′ parallels to the axes.

Since the tangent at R is parallel to PV, RR′ bisects PV at M.

Therefore Q is on RR′.

(1) Reciprocating with P as centre, the conic reciprocates into a parabola, and the extremities of any chord which subtends a right angle at P into tangents at right angles. These meet on the directrix of the parabola. Therefore all such chords pass through a point.

The normal and RR′ are two such chords. Therefore Q is the point.

(2) Draw PH parallel to RR′. R′G is the normal at R′, and OM is at right angles to PV. Therefore OM and R′G are parallel. Also

$$CH = 2CN, \quad PH = CP;$$

$$\therefore \quad OQ : OM = QG : PG = CQ : PH = CG : GH$$
$$= CN - NG : CN + NG = AC^2 - BC^2 : AC^2 + BC^2.$$

4927. (Sir R. S. BALL, F.R.S.)—If k be the constant term in the equation of a surface, and $\Delta = 0$ the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x(d/dk)}\Delta = 0$?

Solution by Rev. J. CULLEN, B.A.; Prof. KRISHMACHANDRA DE; *and others.*

Let $$\psi(\xi, \eta, \zeta) = k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

be the surface. Then, since Δ is a function of k (ϕk, say), we have $e^{-x(d/dk)}\Delta = e^{-x(d/dk)}\phi(k) = \phi(k-x)$. Hence $\phi(k-x) = 0$ is clearly the condition that the surface $\psi(\xi, \eta, \zeta) = k-x$ should also pass through the point common to the three others. Therefore, if a, &c., be the roots of $\phi(k-x) = 0$ (*i.e.*, $e^{-x(d/dk)}\Delta = 0$), we obtain a set of surfaces similar and similarly situated with respect to (1) passing through the given point by writing $k-a$, &c., for k.

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13298. (R. F. Davis, M.A.)—Employing the usual notation for a triangle, note the following expressions

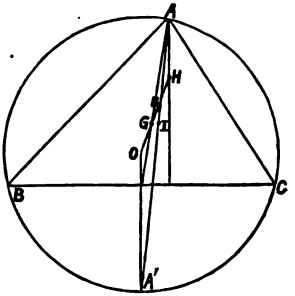
$$OI^2 = R^2 - 2Rr = R(R-2r), \quad 4IN^2 = (R-2r)^2,$$
$$OH^2 = 9R^2 + 8Rr + 2r^2 - 2s^2 = (R-2r)(9R+26r) - 2\Delta^2,$$
$$9IG^2 = 5r^2 - 16Rr + s^2 = \Delta^2 - 16r(R-2r),$$
$$IH^2 = 4R^2 + 4Rr + 3r^2 - s^2 = 4(R-2r)(R+3r) - \Delta^2,$$
$$\&c., \quad \&c.,$$

where $\Delta^2 = s^2 - 27r^2$; and show generally that any symmetrical function of the sides or angles of a triangle may be expressed in terms of s, R, r [In case of equilateral triangle $R = 2r$, and $\Delta^2 = 0$.]

Solution by H. W. Curjel, M.A.; Prof. Krishnamacharry; *and others.*

$$a+b+c = 2s,$$
$$bc+ca+ab = s^2 + r^2 + 4Rr,$$
$$abc = 4Rrs;$$

therefore any symmetrical function of the sides and angles of a triangle can be expressed in terms of s, R, r.



Again, O, G, N, H are collinear, and

$$ON = NH \quad \text{and} \quad HG = 2GO.$$

Since the in-circle touches the N.P.-circle (radius $\frac{1}{2}R$),

$$\therefore \quad 4IN^2 = (R-2r)^2.$$

If AI meets the circum-circle in A', then

$$R^2 - OI^2 = AI \cdot IA'$$
$$= AI \cdot CA' = \{r/(\sin\tfrac{1}{2}A)\} \times 2R\sin\tfrac{1}{2}A = 2Rr;$$
$$\therefore \quad OI^2 = R^2 - 2Rr.$$
$$OH^2 = (R\sin A - 2R\sin B\cos C)^2 + (2R\cos B\cos C - R\cos A)^2$$
$$= R^2(1 - 8\cos A\cos B\cos C)$$
$$= R^2(1 + 8 - 4\sin^2 A - 4\sin^2 B - 4\sin^2 C)$$
$$= 9R^2 - \{(16S^2)/(a^2b^2c^2)\}(a^2+b^2+c^2)R^2$$
$$= 9R^2 + 8Rr + 2r^2 - 2s^2 = (R-2r)(9R+26r) - 2\Delta^2.$$
$$OI^2 + HI^2 = \tfrac{1}{2}(HO^2) + 2IN^2 = 5R^2 + 2Rr + 3r^2 - s^2;$$
$$\therefore \quad IH^2 = 4R^2 + 4Rr + 3r^2 - s^2 = 4(R-2r)(R+3r) - \Delta^2.$$
$$2OI^2 + IH^2 = 3IG^2 + 2OG^2 + GH^2 = 3IG^2 + \tfrac{3}{2}OH^2;$$
$$\therefore \quad 9IG^2 = 6IO^2 + 3IH^2 - 2HO^2 = 5r^2 - 16Rr + s^2 = \Delta^2 - 16r(R-2r).$$

13299. (P. W. Flood.)—Between a tangent drawn at one end of the segment of a circle and the chord produced draw another tangent divided by the point of contact into two segments that shall be in a given ratio.

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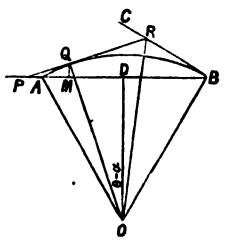
Solution by D. BIDDLE, Rev. H. J. STEWART, *and others.*



Let AB be the base of the given segment. Bisect it in D, and make AM/MD = m, the given ratio. Draw MQ at right angles to AB, and through Q, the point of contact, draw PR, the required tangent.

Let AOB = 2α, QOB = θ, OA = unity.

Then we have

$$\frac{PQ}{QR} = \frac{\{1-\cos\alpha\sec(\theta-\alpha)\}\cot(\theta-\alpha)}{\tan\frac{1}{2}\theta} = m,$$

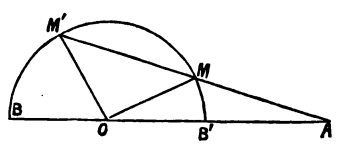
whence, taking $\tan\frac{1}{2}\theta = \sin\theta/(1+\cos\theta)$,

we obtain $(m+1)\cos\alpha\sin\theta-(m+1)\sin\alpha\cos\theta-\sin\alpha = 0$;

$$\therefore \sin(\theta-\alpha) = \sin\alpha/(m+1) = MD.$$

13296. (Rev. T. WIGGINS, B.A.)—Given a circle of radius R, on any diameter BB′ produced take a point A, such that the distance from A to O, the centre, is equal to a. Draw a secant AMM′ making with AO an angle θ. Let β, γ be the angles which the radii OM, OM′ make with the diameter BB′. It is required (1) to calculate $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ in terms of R, a, θ; (2) to show that the product $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ is constant, whatever be the value of θ; (3) to find the minimum value of the sum $\tan^2\frac{1}{2}\beta + \tan^2\frac{1}{2}\gamma$.

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; *and others.*



$$\sin OMA = \sin OM'A = (a\sin\theta)/R,$$

$$-\cos OMA = \cos OM'A = (1/R)(R^2-a^2\sin^2\theta)^{\frac{1}{2}};$$

$$\therefore \cos\beta = \sin M\sin\theta-\cos M\cos\theta = (1/R)\{a\sin^2\theta+\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}\},$$

and $$\cos\gamma = (1/R)\{a\sin^2\theta-\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}\}.$$

Hence we have

$$\tan^2\frac{\beta}{2} = \frac{1-\cos\beta}{1+\cos\beta} = \frac{R-a\sin^2\theta-\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}}{R+a\sin^2\theta-\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}},$$

and $$\tan^2\frac{\gamma}{2} = \frac{1-\cos\gamma}{1+\cos\gamma} = \frac{R-a\sin^2\theta+\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}}{R+a\sin^2\theta-\cos\theta(R^2-a^2\sin^2\theta)^{\frac{1}{2}}};$$

$$\therefore \tan^2\tfrac{1}{2}\beta\tan^2\tfrac{1}{2}\gamma = \{(a-R)/(a+R)\}^2;$$

$$\therefore \tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma = (a-R)/(a+R),$$

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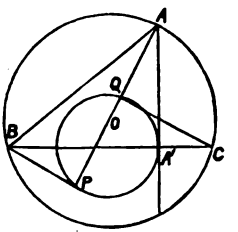
and $\tan^2\frac{1}{2}\beta + \tan^2\frac{1}{2}\gamma = \{2R^2(1+2\cot^2\theta) - 2a^2\} / \{(R+a)^2\}$,
and is a minimum when $\tan\frac{1}{2}\beta = \tan\frac{1}{2}\gamma$;

$\therefore$ minimum value $= \{2(a-R)\}/(a+R)$.

13281. (Professor BERNÉS.)—Dans un triangle ABC, le pied de la hauteur abaissée de A étant A′, et P, Q désignant les projections de B, C sur le rayon AO du cercle ABC, le cercle A′PQ a pour centre C milieu de BC.

Solution by W. E. JEFFARES, B.A.; H. W. CURJEL, M.A.; *and others.*



△s AA′C, APB are similar, and also △s AQC, AA′B.

$\therefore$ $AA'^2 = AQ.AP$;

therefore AA′ touches ⊙A′PQ; therefore centre of ⊙A′PB is in BC;

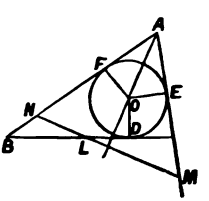
But it is also in the line bisecting PQ at right angles, which also bisects BC.

Therefore centre of ⊙A′PQ is at the middle point of BC.

13258. (Professor SIVERLY.)—Of all cones which develop an ellipsoid and have their bases in the tangent plane at a given point P, and are of the same altitude, that is the least which has its vertex in the diameter through P; and of all which have their vertices in this diameter, that is the least whose axis is twice this diameter.

13274. (Rev. T. WIGGINS, B.A.)—Inscribe the maximum ellipsoid in a cone of given volume.

Solution by H. W. CURJEL, M.A.; Professor BEYENS; *and others.*



Since any ellipsoid may be projected orthogonally into a sphere, the theorem is true for an ellipsoid if it is true for a sphere.

Consider a cone enveloping a sphere, and draw a plane through the centre O of the sphere and the vertex A of the cone, perpendicular to the base, and cutting the cone in △ABC and the sphere in ⊙DEF. Through the middle point L of BC draw NLM perpendicular to AO, cutting AB, AC in N, M.

Then CB = major axis of base and $NL.LM = (\text{semi-minor axis})^2$.

Let $\angle DOA = \pi - \theta$ and $AO = k\sec\theta$, $OD = r$ and $\angle AOF = \phi$;

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then $\quad BOF = \frac{1}{2}(\pi-\phi+\theta), \quad COE = \frac{1}{2}(\pi-\phi-\theta),$

$$BC = BF + CE = r(\tan BOF + \tan COE) = \frac{2r(k^2-r^2\cos^2\theta)^{\frac{1}{2}}}{\cos\theta(k-r)},$$

$$NL \cdot LM = \frac{a^2}{4}\frac{\sin(\theta+\phi)\sin(\phi-\theta)}{\sin^2\phi} = \frac{a^2(\cos^2\theta-\cos^2\phi)}{4\sin^2\phi}$$

$$= \frac{a^2\cos^2\theta(k^2-r^2)}{4(k^2-r^2\cos^2\theta)} = r^2\frac{k+r}{k-r};$$

$$\therefore \quad \text{area of base} = \frac{\pi r^2(k^2-r^2\cos^2\theta)^{\frac{1}{2}}}{\cos\theta(k-r)}\left(\frac{k+r}{k-r}\right)^{\frac{1}{2}},$$

which is clearly a minimum when $\theta = 0$.

Hence first part of theorem.

When $\theta = 0$, $\quad$ volume $= \frac{\pi}{3}\frac{r^2(k+r)^2}{k-r}$;

$\therefore$ for minimum volume, $2(k-r) = k+r$, *i.e.*, $k = 3r$;

$\therefore$ height of cone = 2 diameter of sphere.

Hence second part of theorem.

(13274.) It is clear that, in the last case, the polar plane of the vertex is parallel to the base, and divides the altitude in the ratio 2 : 1. Hence the maximum ellipsoid inscribed in a cone touches the base in its centre, and the conic of contact with the cone is in a plane parallel to the base dividing the altitude in the ratio 2 : 1, and its volume is $\frac{4}{9}$ that of cone.

The points of contact of the ellipsoid may also be constructed by joining the vertex or any point on the circumference of the base to the centre of gravity of the cone (considered as solid), and producing the straight line till it meets the surface.

The centre of the ellipsoid is at the centre of gravity of the cone.

13222. (Professor Coondoo.)—If on a straight line of length $a+b+e$ there be measured at random two lengths a, b, the chance of their having a common part not greater than d is $\frac{(c+d)^2-c^2}{(c+a)(c+b)}$, $(d < a$ or $b)$; the chance of their not having a common part greater than d is $\frac{(c+d)^2}{(c+a)(c+b)}$, and the chance of the smaller b lying altogether within the larger a is $\frac{a-b}{a+c}$.

Solution by H. W. Curjel, M.A.; Prof. Mukhopadhyay; *and others.*

Let x be the common part (when x is negative, the segments are separated by distance $-x$).

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Chance that there is a common part $\not> d$

$$= \frac{\int_0^d 2(c+x)\,dx}{(c+a)(b+c)} = \frac{(c+d)^2-c^2}{(c+a)(c+b)}.$$

Chance that there is not a common part $> d$

$$= \frac{\int_{-c}^d 2(c+x)\,dx}{(c+a)(b+c)} = \frac{(c+d)^2}{(c+a)(b+c)}.$$

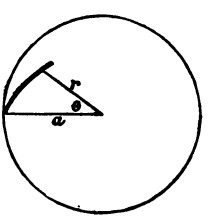
Chance that b lies within $a = \frac{a-b}{a+c}$.

For the favourable range $= a-b$, and whole range $= a+c$.

13261. (Professor LILLEY.)—A drawbridge, a feet in length, moves uniformly about a centre axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its centre, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when $a = 320$ and $n = 2$.

Solution by Rev. J. CULLEN, B.A.; Professor NATH COONDOO; *and others.*

Let v be the velocity of the man relatively to the bridge, and v' that of the end of the bridge. Then we have



$$2a_1 = a, \quad a_1 - r = vt, \quad a_1\theta = v't;$$

also $v/v' = 2a_1/2n\pi a_1$, or $v' = n\pi v$;

$$\therefore \quad r = a_1(n\pi - \theta)/n\pi,$$

the absolute path of the man.

If V be his velocity at any instant in this path, we have

$$V = ds/dt = \left\{(rd\theta)^2/dt^2 + (dr/dt)^2\right\}^{\frac{1}{2}}, \quad dr/dt = v, \quad \text{and} \quad a\,d\theta/dt = v';$$

$$\therefore \quad V/v' = \left\{r^2/a_1^2 + 1/n^2\pi^2\right\}^{\frac{1}{2}} = \left\{(n\pi-\theta)^2 + 1\right\}^{\frac{1}{2}}/n\pi.$$

Putting $a = 320$ and $n = 2$, we get

$$r = 160 - 80\,\theta/\pi, \quad V/v' = \left\{(2\pi-\theta)^2 + 1\right\}^{\frac{1}{2}}/2\pi.$$

13265. (C. E. HILLYER.)—Having given a straight line in a circle, construct a second straight line equal to the first, so that the lines through the middle point of the first parallel to the second and through

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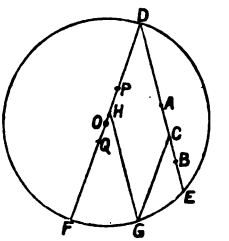
the middle point of the second parallel to the first may meet on the circumference.

Solution by D. BIDDLE; H. W. CURJEL, M.A.; *and others.*



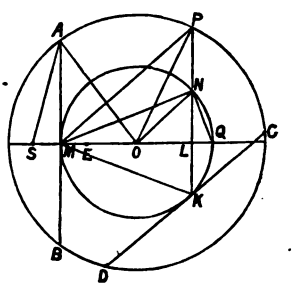
Numerous lines can be drawn fulfilling the conditions, but the following method is of universal application:—

Let AB be the given line, and C its middle point. Produce it both ways to the circumference at D and E. From D, the furthest point from C, draw the diameter DF. Then draw CG, GH parallel to DF, DE, respectively, and on DF make

$$HP = HQ = AC = BC.$$

PQ fulfils the conditions, and it is evident that, if through P, Q parallels to DE were drawn, innumerable parallels to DF would intersect both within the circle, and the intercepted portions would equally fulfil the conditions. Nor are these all: for DF might be a *chord*, not often less than DE, but not necessarily a diameter.

[Having amended the above question by the substitution of *chord* for *straight line* in both places, the PROPOSER sends the following solution:—Let AB be the given straight line in the given circle whose centre is O.



With centre O describe a circle touching AB at M, its middle point.

In OM or OM produced take a point S such that SA = SO. Bisect OS at E, and take L in EO produced so that $EL^2 = \frac{1}{2}OA^2 + EO^2$.

Through L draw a perpendicular to ML meeting the inner circle in N and K and the outer in P. The tangents to the inner circle at N and K terminated by the outer circle shall be the straight lines required.

Let ML produced meet the inner circle at Q, and let the tangent at K meet the outer circle in C and D.

Then CD = AB. Also $OP^2 = OA^2 = 2OM.OS = QM.OS$.

And, since OS is bisected at E and produced to L,

$$OL.LS = EL^2 - EO^2 = \tfrac{1}{2}OA^2 = OM.OS.$$

$$\therefore\ OL^2 = OL.LS - OL.OS = OM.OS - OL.OS$$
$$= OQ.OS - OL.OS = QL.OS.$$
$$\therefore\ OL^2/OP^2 = QL/QM = QN^2/QM^2.$$

$\therefore\ \angle OMN = OPN$, and O, M, P, N are concyclic.

$$\therefore\ \angle MPN = NOL = 2NML = NMK = NKC.$$

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$\therefore$ MP is parallel to CD, and KP is parallel to AB.
Similarly the tangent at N satisfies the conditions.

Mr. BIDDLE, having seen this solution of the amended question, observes that PM, PK are obviously equal, being sides of an equilateral parallelogram completed by producing AB, CD until they meet. A less intricate solution of the question, therefore, seems possible. Let OA = a, OM = b, ML = x. Then

$$MP^2 = x^2 + a^2 - (x-b)^2 = PK^2 = \left[\left\{a^2-(x-b)^2\right\}^{\frac{1}{2}} + \left\{b^2-(x-b)^2\right\}^{\frac{1}{2}}\right]^2;$$

whence $$x(x-b)^2 = (2b-x)(a^2-b^2+2bx-x^2),$$

and $$x = \left\{\pm(a^4+8a^2b^2)^{\frac{1}{2}} - a^2 + 4b^2\right\}/(4b).$$

This may be further simplified by taking $c = 2b$ and $m = a/(2c)$; whence

$$OL = x - b = m\left\{(a^2+2c^2)^{\frac{1}{2}} - a\right\},$$

which can be readily applied in geometric form to determine the position of L and thence of K. Strange to say, by taking x = OL, a cubic equation results, and the hope is afforded us of finding a ready means of solving cubics irreducible by CARDAN'S method.]

11528. (EDITOR.)—Required a general investigation of the following problem, proposed by the EDITOR many years ago as Quest. 1072, and solved, when reproposed as Quest. 4390, on p. 87 of Vol. XLII.:—"Two gamesters, A and B, play together, A having the power to fix the stakes. Whenever A loses a game, he increases his last stake by a shilling for the next game, and diminishes it by a shilling after each gain. When they leave off playing, A has gained £13; and, had each won the same number of games, A would still, by following this method in regulating his stakes, have won 10 shillings. The first stake having been 30 shillings, find the number of games each won." [This question is reproposed for sake of its relation to "the WELLS Martingale," or "steady and certain system" of winning at Monte Carlo, which is thus set forth in the second leader of the *Daily Telegraph* for April 18th, 1892: "You always play on the colour which turned up last, and you always add one unit after every loss, and risk one less after every time you have won." Under Quest. 4596, a similar problem is solved, on p. 35 of Vol. XXIII. An investigation is desired that shows the relation of the *general* problem to "the WELLS Martingale."]

Solution by H. W. CURJEL, M.A.; Prof. KRISHMACHANDRA DE; *and others*.

On an adjacent gain and loss, A wins a shilling, and the stakes after the gain and loss are the same as if the gain and loss are omitted. Hence the amount won or lost is independent of the order of the gains and losses.

Hence the number of games played = 20.

Let y = number of games B wins, and $x+y$ = number of games A wins.

Then $$x + 2y = 20$$

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and $260 = \frac{1}{2}(20-x) + \{30+29+\ldots+(31-x)\} = \frac{1}{2}(20-x) + \frac{1}{2}\{x(61-x)\}$.

$\therefore\ x^2-60x+500 = 0;\quad \therefore\ x = 10 \text{ or } 50.$

Therefore A wins 15 games and B 5 games.

13224. (Professor KRISHMACHANDRA DE.)—If $\rho_1, \rho_2, \ldots \rho_n$ be the distances of a point P in the plane of a regular polygon of n sides from he vertices, prove that

$$\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\ldots+\frac{1}{\rho_n^2} = \frac{n}{r^2-a^2}\cdot\frac{r^{2n}-a^{2n}}{r^{2n}-2r^n a^n\cos n\theta+a^{2n}},$$

Solution by W. J. GREENSTREET, M.A.; Rev. J. CULLEN, B.A.; *and others.*

If a be the circum-radius of the polygon, r the join of P and the centre O, θ the angle OP makes with the radius to any angular point,

$\rho_1^2 = r^2-2ra\cos\theta+a^2,\quad \rho_2^2 = r^2-2ra\cos\{\theta+(2\pi/n)\}+a^2,\quad$ &c.

Now $r^{2n}-2r^n a^n\cos n\theta+a^{2n}$

$$= (r^2-2ra\cos\theta+a^2)\left[r^2-2ra\cos\{\theta+(2\pi/n)\}+a^2\right]\ldots.$$

Taking logs, and differentiating with respect to r,

$$n\cdot\frac{2r^{2n-1}-2r^{n-1}a^n\cos n\theta}{r^{2n}-2r^n a^n\cos n\theta+a^{2n}} = \frac{2r-2a\cos\theta}{r^2-2ra\cos\theta+a^2}+\frac{2r-2a\cos\{\theta+(2\pi/n)\}}{r^2-2ra\cos\{\theta+(2\pi/n)\}+a^2}+\ldots.$$

Therefore, multiplying by r and arranging by subtracting unity from each right-hand fraction and n from the left-hand,

$$n\cdot\frac{r^{2n}-a^{2n}}{r^{2n}-2r^n a^n\cos n\theta+a^{2n}} = \frac{r^2-a^2}{r^2-2ra\cos\theta+a^2}+\frac{r^2-a^2}{r^2-2ra\cos\{\theta+(2\pi/n)\}+a^2}+\ldots.$$

$$= (r^2-a^2)\Sigma\frac{1}{\rho_n^2}.\qquad \text{Therefore &c.}$$

13292. (F. H. JACKSON.)—Prove that

$$1+n\,.\,x_1y_{-1}+\frac{n\,.\,n-1}{2!}x_2y_{-2}+\ldots = \frac{(y)_{-n}}{(x+y)_{-n}},$$

where
$$x_n = \frac{\Pi(x)}{\Pi(x-n)} \qquad \ldots\ldots(1),$$

analogous to the binomial expansion

$$1+n^1xy^{-1}+\frac{n\,.\,n-1}{2!}x^2y^{-2}+\ldots = \frac{(y)^{-n}}{(x+y)^{-n}} \qquad \ldots\ldots(2),$$

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No. (1) being a variation of VANDERMONDE's theorem, as (2) is of the binomial theorem; n is not necessarily an integer in (1).

Solution by H. W. CURJEL, M.A.; Professor NATH COONDOO; *and others.*

Comparing the series (1) with the hypogeometric series, we see that it is convergent if $x+y+n+1>0$. Let $x=-z$, $y=-m$.

$$\text{Series (1)} = 1+\frac{m\,.\,z}{1\,(y+1)}+\frac{m\,(m+1)}{1\,.\,2}\,\frac{z\,(z+1)}{(y+1)(y+2)}+\ldots$$
$$=\frac{\Pi\,(y)\;\Pi\,(y-m-z)}{\Pi\,(y-m)\;\Pi\,(y-z)}$$

[FORSYTH, *Differential Equations*, § 126, or Solution to Quest. 13215]

$$=\frac{\Pi\,(y)\;\Pi\,(x+y+n)}{\Pi\,(y+n)\;\Pi\,(y+x)}=\frac{(y)_{-n}}{(x+y)_{-n}}.$$

Series (2) is convergent if $y>x$, or if $x=y$, $n>0$, and is equal to

$$\left(1+\frac{x}{y}\right)^n=\frac{(x+y)^n}{y^n}=\frac{y^{-n}}{(x+y)^{-n}}.$$

13288. (H. MACCOLL, B.A.)—Prove the following formulæ in symbolic logic:—

(1) $(\alpha+\beta)_x : \alpha_x+\beta_x : (\alpha\beta)_x$, (2) $x_{\alpha\beta} : x_\alpha+x_\beta : x_{\alpha+\beta}$,

(3) $(\alpha+\beta)_{o x}=\alpha_{o x}+\beta_{o x}$, (4) $x_\alpha x_{o\beta} : \alpha_{o\beta}$,

(5) $x/\alpha\,.\,x/o\beta : \alpha/o\beta$, (6) $\alpha/\beta=\alpha_\beta\epsilon_{o\beta}$, (7) $\alpha_\beta=(\epsilon : \beta^\alpha)$.

Defs.—A : B : C is short for (A : B)(B : C); $\alpha_{o\beta}$ is the denial of α_β; $\alpha/o\beta$ is the denial of α/β; the symbol ϵ denotes a statement *always true*, so that ϵ_α asserts that α is *always true*, and $\epsilon_{o\alpha}$ asserts that α is *not* always true. [See Quest. 13234.]

Solution by H. W. CURJEL, M.A.; Prof. KRISHMACHANDRA DE; *and others.*

Using the same notation as in the solution to Quest. 13234,

(1) Given $(\alpha+\beta)_x$, *i.e.*, $A+B+X=X$; therefore $A+X=X$ and $B+X=X$, *i.e.*, $\alpha_x\beta_x$; therefore $\alpha_x+\beta_x$, *i.e.*, $A+X=X$ or $B+X=X$; therefore $AB+X=X$, *i.e.*, $(\alpha\beta)_x$.

(2) Given $x_{\alpha\beta}$, *i.e.*, $X+AB=AB$; therefore

$$X+A=X+AB+A\bar{B}=AB+A\bar{B}=A,$$
$$X+B=X+AB+\bar{A}B=AB+\bar{A}B=B,$$

i.e., $x_\alpha x_\beta$, therefore $x_\alpha+x_\beta$, *i.e.*, either $X+A=A$ or $X+B=B$; therefore $X+A+B=A+B$, *i.e.*, $x_{\alpha+\beta}$.

(3) If $A+B+X\neq X$, either $A+X\neq X$ or $B+X\neq X$, and, if $A+X\neq X$ or $B+X\neq X$, then $A+B+X\neq X$, *i.e.*,

$$(\alpha+\beta)_{ox}=\alpha_{ox}+\beta_{ox}.$$

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(4) Given $x_\alpha x_{\alpha\beta}$, *i.e.*, $AX = X$ and $BX \neq X$, $XAB = XB \neq X \neq AX$; therefore $AB \neq A$, *i.e.*, $a_{\alpha\beta}$.

(5) Given x/a, $x/\alpha\beta$, *i.e.*, $XA = X$ and $A \neq 1$, and $XB \neq X$ or $B = 1$, we have $B = 1$ or $AB \neq A$ by (4), *i.e.*, $a/\alpha\beta$.

(6) a/β is the same as $A + B = B$ and $B \neq 1$, *i.e.*, $a_\beta \cdot \epsilon_{\alpha\beta}$, for $E = 1$.

(7) If α^α, *i.e.*, $A + B = B$, then $B + \bar{A} = A + B + \bar{A} = 1 + B = 1$, *i.e.*, β^α, *i.e.*, $\epsilon : \beta^\alpha$.

If $\epsilon : \beta^\alpha$, *i.e.*, $B + \bar{A} = 1$, $AB + A\bar{A} = A$; therefore $AB = A$, *i.e.*, a_β.

12868. (H. J. Woodall, A.R.C.S.)—Elucidate the following extract from the "Bemerkungen," which Gauss' Editor (Schering) has collected at the end of Vol. II. of the *Werke* (p. 523). "Es bezeichnen darin [2], [5], [13], ... (18), (57), (239), ... die Bögen der Cotangenten 1, 2, $\frac{3}{4}$, ... 18, 57, 239, ... Mit Hülfe der Tafeln ist durch Zurlegung von $18 + i$, $57 + i$, $239 + i$ in ihre complexe Primfactoren

$$(18) = 2[2] - 2[5] - [13], \quad (57) = -[2] + 3[5] - [13],$$
$$(239) = 3[2] - 4[13] \text{ gefunden.}"$$

The tables referred to are explained thus:—"Die Tafeln *zur Cyklotechnie* geben für 2452 Zahlen von der form $a^2 + 1$, $a^2 + 4$, $a^2 + 9$, ... $a^2 + 81$ die sämmtlichen ungeraden Primtheiler p neben den zugehörigen a und zwar in solchen Fällen, wo die Primtheiler alle unter 200 liegen, nur dann werden $a^2 + 1$ u.s.f. zerlegen genannt."

Extract from table $a^2 + 1$.

18	5 . 5 . 13
57	5 . 5 . 5 . 13
239	13 . 13 . 13 . 13

Solution by Professor Sanjána; H. W. Curjel, M.A.; *and others.*

$[a^2 + b^2] = \text{arc cot } a/b$, if $a^2 + b^2$ is a prime, and $[a^2 + 1] = (a) = \text{arc cot } a$, if $a^2 + 1$ is a prime. From "table $a^2 + 1$"

$$18^2 + 1 = 5 \times 5 \times 13 = (2^2 + 1)^2 (3^2 + 2^2), \quad 57^2 + 1 = (1^2 + 1^2)(2^2 + 1)^3 (3^2 + 2^2),$$
$$239^2 + 1 = (1^2 + 1^2)(3^2 + 2^2)^4;$$

therefore $18 + i = (2 \pm i)^2 (3 \pm 2i)$, $57 + i = (1 \pm i)(2 \pm i)^3 (3 \pm 2i)$,

$$239 + i = (1 \pm i)(3 \pm 2i)^4,$$

in each case possibly with factor $\pm i$ in addition. We easily verify the signs and get

$$18 + i = i(2 - i)^2 (3 - 2i), \quad 57 + i = (2 + i)^3 (1 - i)(3 - 2i),$$
$$239 + i = i(1 + i)(3 - 2i)^4.$$

Hence, with the help of Demoivre's theorem, we get

$$(18) = 2[2] - 2[5] - [13], \quad (57) = -[2] + 3[5] - [13],$$
$$(239) = 3[2] - 4[13].$$

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APPENDIX.

UNSOLVED QUESTIONS.

4953. (Editor.)—A King is placed at random on a clear chess board, and then similarly (1) a Bishop, or (2) a Rook; find, in each case, the chance that the King is in check so as to be unable to take the attacking piece; and find also (3) the chance of check, with or without the power of taking, for any combination of two or three of the pieces. [If we estimate the powers of the pieces (α) by the chances of *simple* check, as investigated in the Solution of Question 3314, *Reprint*, Vol. XV., pp. 50, 51, in January, 1871; (β) by the chances of *safe* check, as shown in an interesting paper by H. M. Taylor in the *Philosophical Magazine* for March, 1876; (γ) by the *results* given in the *Berliner Schachzeitung*; we have the relative values of the Knight, Bishop, Rook, Queen, as

(α) 3 : 5 : 8 : 13; (β) 3 : 3¼ : 6 : 9¼; (γ) 3 : 3½ : 4½ : 9½.

4959. (S. Watson.)—Two circles (each radius r) are described on the semi-minor axes of an ellipse as diameters, and two others (each radius r) having their centres on the major axis to touch the former two, and the ellipse A is the area of the rhombus formed by joining the centres of the four circles, and A′ the area of that formed by tangents at the points of contact of the circles; then

$$\frac{r'}{r} = 2e, \quad \text{and} \quad \frac{R}{R'} = 1 + e.$$

4963. (Professor Genese, M.A.) — It is objected to the use of negative values for r in polar coordinates, (1) that the value of θ is altered, (2) that the trigonometrical functions of θ are *expressly* defined with r *positive*. Required logical answers to these. Also an explanation of the fact that in tracing $a = r\cos\theta$ we have to go over every point *twice*.

4970. (Dr. Artemas Martin.)—Two radii are drawn at random in a given semicircle, and then two points are taken at random, one in each of the radii; find the average distance between the points.

4986. (W. S. B. Woolhouse, F.R.A.S.)—Determine a general expression for the average of the *positive* values of the linear function $h + A_1x_1 + A_2x_2 + A_3x_3 + \&c.$, when the variables x_1, x_2, x_3, &c., take all values between given limits, the constants h, A_1, A_2, A_3 having each of them a fixed sign, either positive or negative.

4989. (J. J. Walker, M.A.)—Deduce the form of the Catenary from the consideration that its centre of gravity must be the lowest possible.

4990. (J. L. McKenzie.)—If the equation of a curve of the nth degree

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be of the form $k\phi = a^2\psi$, where a is linear, and if m be the degree of the first pedal: prove (1) that the pedal has one real asymptote, which is perpendicular to a; (2) that it passes $\frac{1}{2}(m-1)$ times through the circular points; (3) that m is not greater than $2n(n-1)-1$, and is diminished by 4 for every double point on the original curve, and by 6 for every cusp.

4991. (G. A. Ogilvie.)—Show (1) that there is only one point the polars of which with respect to four conics form a parallelogram, and (2) find the area of the parallelogram for any four conics.

4998. (Colonel Clarke, C.B., F.R.S.)—In Question 4736, the rod being allowed to start from an initial value a of θ, determine the time the system takes to come to rest. Also determine this, when p is very much greater than m. [Question 4736, proposed by C. Leudesdorf, and solved in Vol. xxv., p. 67, is as follows: "A uniform heavy rod AB can move freely in a vertical plane about a horizontal fixed axis passing through A. C is the position of B when the rod hangs vertically downwards. BCP is an inextensible string attached to the rod at B, and which, passing through a smooth ring at C, carries a weight P at its other extremity. If m, p be the masses of the rod and the weight respectively, show that, if the rod be slightly displaced from its position of equilibrium, it will oscillate if $m > 2p$, and that the time of an oscillation is to the time of an oscillation when the weight is removed as $m : \{(m-2p)(m+3p)\}^{\frac{1}{2}}$.]

5003. (C. H. Hinton.)—If on the sphere of projection P_1, P_2, P_3 are the poles of three faces of an anorthic crystal, the symbols being $h_1k_1l_1$, $h_2k_2l_2$, $h_3k_3l_3$ respectively; and A, B, C are the poles 100, 010, 001, X, Y, Z the points where the axes cut the sphere; show (1) that the following is the expression for $\tan P_2P_1P_3$,

$$-\left[\frac{h_1^2}{a^2}\sin^2 x + \frac{k_1^2}{b^2}\sin^2 y + \frac{l_1^2}{c^2}\sin^2 z + \frac{2h_1k_1}{ab}\sin x \sin y \cos Z + \frac{2k_1l_1}{bc}\sin y \sin z \cos X + \frac{2l_1h_1}{ca}\sin z \sin x \cos Y\right]^{\frac{1}{2}} \div L_1 + L_2 + L_3 + \cos x\, M_1 + \cos y\, M_2 + \cos z\, M_3;$$

the whole multiplied by $\frac{1}{abc}\begin{vmatrix} h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \\ h_3 & k_3 & l_3 \end{vmatrix}$

where $$L_1 = \frac{h_1k_2 - h_2k_1}{ab} \times \frac{h_2k_3 - h_3k_2}{ab},$$

$$M_1 = \frac{k_1l_2 - k_2l_1}{bc} \cdot \frac{l_2h_3 - l_3h_2}{ca} + \frac{l_1h_2 - l_2h_1}{ca} \cdot \frac{k_2l_3 - k_3l_2}{bc},$$

and L_2, L_3, M_2, M_3 are obtained from L_1, M_1 respectively by changing each $\frac{h}{a}$ into $\frac{k}{b}$, and so on. Also (2) give a similar general expression for $\tan P_1P_2$.

5012. (Professor Sylvester, F.R.S.) — In a jointed parallelogram ABCD, the side AD, which is equal in length to AB, becomes fixed: prove (1) that, in general, two points may be found, one in BC, and the other in BC produced beyond C, each of which shall move in a circular orbit; and find (2) the condition in order that a point in BC shall move in the same circular orbit as B.

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5014. (Professor Clifford.)—Two planes A, B are said to have an (x, y) correspondence, when to every point on the plane A correspond y points on the plane B, and to every point on B correspond x points on A. On each plane there is in general a locus of points, two of whose correspondents coincide; this is called the *cross-curve*. (Uebergangscurbe: Clebsch in *Math. Annalen*.) On each plane there is also a locus of these united correspondents; this is called the *node-curve*.

1. If a curve touch the cross-curve in either plane, the corresponding curve in the other plane will have a node lying on the node-curve in that plane.

2. The correspondence may be represented as a (1, 1) correspondence of two multiple planes A′, B′; A′ consisting of y sheets and B′ of x sheets, which are connected together along the cross-curves.

3. In a $(1, y)$ correspondence, in which to straight lines in the plane A correspond curves of deficiency p in the plane B, the order of the cross-curve in A is $= 2(y+p-1)$.

5016. (Professor Crofton, F.R.S.)—If there be any closed convex boundary of length L, and θ be its apparent magnitude at any point (x, y), and if an endless string of length Y be passed round it, and kept stretched by a moving pencil which thus traces out an external boundary; prove that the value of the integral $\iint \sin\theta\, dx\, dy$ extended over the annulus between the two boundaries is $\mathrm{L}(\mathrm{Y}-\mathrm{L})$. [If the given curve be an ellipse, the outer one, as is known, is a confocal ellipse.]

5019. (J. J. Walker, M.A.)—If A be the refracting angle of a prism, and θ, ϕ the differences between the angles which the part of the ray within the prism makes with the normals to the faces, and between the deviations at incidence and emergence respectively, while D is the total deviation; prove (1) that $\sin\frac{1}{2}(\mathrm{A}+\mathrm{D}) = \dfrac{m\cos\theta\,\sin\frac{1}{2}\mathrm{A}}{\cos(\theta+\phi)}$, and thence (2) deduce the position of minimum deviation.

5021. (Professor Hudson, M.A.)—In a fluid at rest under the action of forces of which $\mathrm{X}\delta m$, $\mathrm{Y}\delta m$, $\mathrm{Z}\delta m$ are the components parallel to the axes acting upon a small mass δm, the locus of points about which the greatest variation of pressure is a maximum or a minimum is determined by

$$\frac{\mathrm{X}\frac{d\rho\mathrm{X}}{dx}+\mathrm{Y}\frac{d\rho\mathrm{X}}{dy}+\mathrm{Z}\frac{d\rho\mathrm{X}}{dz}}{\mathrm{X}} = \frac{\mathrm{X}\frac{d\rho\mathrm{Y}}{dx}+\mathrm{Y}\frac{d\rho\mathrm{Y}}{dy}+\mathrm{Z}\frac{d\rho\mathrm{Y}}{dz}}{\mathrm{Y}} = \frac{\mathrm{X}\frac{d\rho\mathrm{Z}}{dx}+\mathrm{Y}\frac{d\rho\mathrm{Z}}{dy}+\mathrm{Z}\frac{d\rho\mathrm{Z}}{dz}}{\mathrm{Z}},$$

ρ being the density at x, y, z.

5022. (Rev. E. Hill, M.A.)—A cylindrical thin rigid surface contains fluid at a constant pressure P; show (1) that the tension at any point is given by $\dfrac{d^2\mathrm{T}}{d\phi^2}+\mathrm{T}=\mathrm{P}$; and hence (2) that, if such a circular cylinder

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have a crack through it down a generating line, the tension opposite the crack is 2P × radius, and that the mutual action, normal to the surface, of the surfaces on each side of a generating line is a maximum 90° from the crack.

5024. (J. Griffiths, M.A.)—If P be the point of intersection of the perpendiculars, and O the centre of the circumscribing circle of any plane triangle; prove that (1) the director of any conic touching the sides and passing through O has contacts of a similar species with the nine-point and circumscribing circles; and (2) the directors of the four conics which touch the sides and pass through the points P, O will touch the nine-point circle at its points of contact with the inscribed and escribed circles respectively.

5027. (T. Cotterill, M.A.)—If a, b, c, d be four points on a cubic K having the same tangential n, prove (1) that a cubic will pass through these five points and the three intersections of a line L with K, having a node at n, the intersections of the conic (n, a, b, c, d) and L being points on the nodal tangents; and (2) find linearly the remaining points of the nodal cubic on the six lines through a, b, c, d.

5028. (M. Hermite.)—Prove that (1) the Jacobian of two binary quantics cannot have a less number of real linear factors than the difference between the number of such in the two quantics themselves separately taken; and (2) if F, ϕ, ψ are three binary quantics, the number of real linear factors in F cannot exceed the number of such in

$$(\phi x + \psi y)\left(\phi \frac{d}{dy} - \psi \frac{d}{dx}\right)\mathrm{F}.$$

5036. (N'Importe.)—Show (1) how to obtain the arrangement of squares in Professor Sylvester's "Anallagmatic Chessboard," given on a large folding diagram at the end of Vol. x. of the *Reprint;* and (2) prove that such an "anallagmatic" square cannot be made also "isochromatic," unless the number of squares in each side is an *even* power of 2.

5040. (Professor Wolstenholme.)—Through the foci S, H of a rectangular hyperbola is drawn a circle meeting the curve in P, P′ (on the same side of the axis), and chords P′A′Q′, P′CR, P′AQ of the circle are drawn through the centre and vertices of the hyperbola: then the chord QQ′ and the tangents to the circle at R, P will meet on the axis, also the chord QQ′ and the tangent at P will each be equally inclined to the focal distance of R. (The loci of R, Q, Q′ are obvious, being the inverses of the hyperbola with respect to its centre and vertices.) The points P, Q, Q′ are all such that the distance from R is a mean proportional between the distances from S, H.

5041. (Professor Crofton, F.R.S.)—If a bicircular quartic pass through four fixed points on a circle, and have one focus at the centre of the circle, the locus of its three remaining foci consists of the two circular cubics whose foci are the four given points [the same locus as in Questions 1990 and 2332, Vols. VI., VII.]. If the given focus be not at the centre of the circle, the locus of the three others will generally be the two bicircular quartics passing through that focus, and having the four given points as foci.

5042. (Editor.)—In an ellipsoid whose principal semi-axes are a, b, c,

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if u_1 be the sum of the nth powers of the conjugate semi-axes, and u_2 a like sum of semi-axes that are at right angles to one another; show that the minima values of u_1, u_2 are respectively

$$3^{1-\frac{1}{2}n}(a^2+b^2+c^2)^{\frac{1}{2}n}, \quad 3^{1+\frac{1}{2}n}(a^{-2}+b^{-2}+c^{-2})^{-\frac{1}{2}n}.$$

5043. (T. Cotterill, M.A.)—1. If capital letters denote points in space, and A_1A_2 are harmonic conjugates to DA, B_1B_2 to DB, and C_1C_2 to DC, then the four pairs of planes such as $A_1B_1C_1$ and $A_2B_2C_2$, intersect in four lines in the plane ABC, and ABC is their diagonal triangle.

2. The polar quadric of D to any cubic surface through D, and any three such pairs of points as A_1A_2, B_1B_2, C_1C_2, passes through ABCD. Explain what this becomes, when A, B coincide, and when A, B, C coincide. What must be the nature of a surface of the nth order, that the same theorem may apply ?

5045. (Professor Hudson, M.A.)—A quantity of air in a cylinder, confined by a piston of given weight, receives a certain amount of heat Q; prove that the increment of volume is $Q\kappa a \div p\sigma$, where σ is the specific heat of air at constant pressure, and p, κ, a have their usual meanings.

5046. (S. A. Renshaw.)—If abc be a triangle inscribed in a conic, and one of its sides, as ac, be produced to meet the directrix in p_2, and Fp_2 be joined, F being the focus, and Fk be drawn at right angles thereto, and meeting the curve in k; then, if kb be joined and produced to meet the directrix in p_4, also if ab, cb be produced to meet the directrix in p_1, p_3 and Fp_1, Fp_3, Fp_4 be joined, the angle p_1Fp_3 is bisected by Fp_4, and $p_2F.p_3F = p_1p_4.p_4p_3 + Fp_4^2$. Similarly, if k' be the opposite extremity to k of the focal chord kF, and bk' meets the directrix in p_6, then Fp_6 joined will bisect the supplement of the angle p_1Fp_3 and

$$p_1F_1Fp_3 = p_1p_6.p_6p_3 - Fp_6^2.$$

5049. (S. Tebay, B.A.)—Given the head and bung diameters, and the length of a cask, also the length of the staves, which are of continued curvature; find the mean content of the cask.

5051. (Rev. E. Hill, M.A.)—A cylindrical portion of incompressible fluid, of radius a but infinite length, is acted on by no forces except a constant pressure on its outer surface. If a coaxial cylindrical portion, of radius b and length likewise infinite, be annihilated, show that the pressure at a distance r from the centre is instantaneously diminished by the fraction $\dfrac{\log r - \log a}{\log b - \log a}$ of itself.

5057. (Professor Evans, M.A.)—A line is drawn across a given ellipse at random, and two points are taken in the surface of the ellipse at random; find the probability that the two points are in the same segment of the ellipse.

5060. (Rev. T. P. Kirkman, M.A., F.R.S.)—Within a pentagon five points are pricked in a random line; lines, either straight or curved, are drawn at random, each connecting two of the ten points, till three lines at least meet in every point, the lines being all in the upper face of the pentagon, and no one crossing another; find the chance that the figure is a symmetrical polyhedron.

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5074. (Rev. W. Roberts, M.A.)—Let

$$\tan^2\theta = \cos^2\phi + \cos^2\alpha \sin^2\phi \cos\psi + \cos^2\beta \sin^2\phi \sin^2\psi;$$

it is required to find the value of the double definite integral

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{\theta \sin^2\phi \sin\psi \, d\phi \, d\psi}{\tan\theta}.$$

5084. (Professor Ball, M.A.)—Determine a geometrical construction for the snail of a repeating clock.

5085. (Dr. Booth, F.R.S.)—If α and β are the semiaxes of the evolute of an ellipse, show that there is but one length of flexible radius or string that will describe an ellipse, and that the end of any other length but $\frac{\beta^3}{\beta^2-\alpha^2}$ will describe a curve of a higher order.

5087. (Rev. T. P. Kirkman, M.A., F.R.S.) — The thousands of 10-acral 10-edra, and the thousands of 11-acral 10-edra, have each among them a number > 50 and < 60 of solids, none of which is the reflected image of another, having all a symmetry of repetition about one axis only, and not more symmetrical about any bisecting plane than about another. Which of the two numbers < 60 is the greater?

5089. (W. S. B. Woolhouse, F.R.A.S.)—Each set of simultaneous roots of a given linear equation $A_1z_1 + A_2z_2 + \ldots A_nz_n = B$ is arranged in the order of magnitude: find the average value of the root that stands the mth in order.

5093. (S. Watson, M.A.)—Through the angle A of a given triangle ABC, two straight lines are drawn at random, dividing the triangle into three parts. Also three points are taken at random within the same triangle. Find the respective chances, (1) of all the points lying in some one of the three parts; (2) that two points shall lie in some one of the three parts, one in another, and none in the third; and (3) that one point shall lie in each part.

5094. (Camille Jordan.)—Imagine a given ramification of any form whatever containing n branches in all. At any particular node suppose i branches to be attached. Then cutting out this node the ramification will fall to pieces into i different ramifications containing respectively $a_1, a_2, \ldots a_i$ branches where $a_1 + a_2 + \ldots + a_i = n$. (1) Let n be odd; it is required to prove that there will be one, and only one, node for which the largest of the quantities $a_1, a_2, \ldots a_i$ is not greater than the sum of all the others. (2) Let n be even; it is required to prove that there will be no node for which the largest of the numbers $a_1, a_2, \ldots a_i$ is not greater than the sum of all the others; and two, and only two, for which its excess above that sum is unity.

5102. (Rev. G. H. Hopkins, M.A.)—From any point on the surface of a circular cylinder planes are drawn. Find the equation to the surface upon which the foci of all the elliptic sections are placed; and prove that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the Logocyclic Curve.

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5105. (H. McColl.)—A certain mathematician solved a question in probability, and obtained q as his expression for the required chance. Not feeling satisfied, however, that this result was correct, he applied the test of experiment, and found that the event in question happened m times in n trials. Supposing P to be a fair estimate of the probability, independently of the experiment, that the result q was correct, show that, if the experiment be taken into account, the proper estimate is $a \div (a+b)$, in which $a = \mathrm{P}q^m (1+q)^{n-m}$, and $b = \dfrac{m!\,(n-m)!}{(n+1)!}(1-\mathrm{P})$.

Note.—The numerical result can be easily calculated by the aid of Stirling's theorem.

5108. (S. A. Renshaw.)—Show that, if any side of a self-conjugate triangle inscribed in a conic be produced to meet the directrix, the line joining the point of intersection therewith with the focus is at right angles to the line joining the focus with the opposite vertex of the triangle.

5119. (R. Battle, M.A.)—Required the position of that tangent to a given circular segment of which the part between the chord produced through one end and the tangent at the other end shall be the least possible.

5120. (Professor Genese, M.A.)—Express geometrically the condition that the conic through five given points should be (1) a parabola (2) a rectangular hyperbola, (3) of eccentricity e.

5124. (S. A. Renshaw.)—Ten bicyclists are started at the same time at given distances apart (say $a, b, c, \ldots j$ yards from a given mark). The speed at which each rider can travel is also given (say $m, n \ldots$ &c. miles an hour). Now, if the places from which they start are chosen by lottery, find (1) the chances that at a given time after the start the distance between any two will be less than a given distance; and (2) when the sum of the distances between all the pairs will be less than a given distance.

5126. (Dr. Artemas Martin.)—A square hole is cut at random through a sphere; find the average volume removed. [*Vide* Solution of 5644, Vol. LXV., p. 7.]

5128. (The late G. O'Hanlon.)—Find the envelope of a line joining the positions of a planet at opposite periods of its year.

5139. (T. Cotterill, M.A.)—Find (1) the locus of the cusps of a system of parallel curves; and show (2) that, in the neighbourhood of a cusp on a curve or its evolute, we can render visible two cusps and a node of a parallel to the curve by taking an appropriate modulus.

Mr. Cotterill remarks that Zeuthen has given $m(m+2n-5)+\tau$ as the order of the locus of the nodes of the system of parallels to a curve of the order m, class n, with τ bitangents.

5142. (J. Griffiths, M.A.)—If δ_1, δ_2, δ_3 be the distances between the centres of the pairs of given circles S_2S_3, S_3S_1, S_1S_2, where $S_1 = x^2 + y^2 + \ldots = 0$, $S_2 = \ldots = 0$, $S_3 = \ldots = 0$; show that

$$\delta_1 S_1^{\frac{1}{2}} + \delta_2 S_2^{\frac{1}{2}} + \delta_3 S_3^{\frac{1}{2}} = 0$$

gives a unique solution of the problem, to draw a Cartesian which shall have double contact with each of three given circles.

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5150. (Rev. T. P. Kirkman, M.A., F.R.S.) — If $R = m_1e_1 + m_2e_2 + m_3e_3 + \ldots$ be every partition in turn of R into positive multiples of different numbers $e_1e_2e_3\ldots$, and

$$S_R = \Sigma \frac{1}{e_1^{m_1}e_2^{m_2}\ldots \Pi m_1 \Pi m_2 \ldots},$$

in which every partition of R gives a term under Σ; prove that

$$R = S_R + S_{R+1} + S_{R+2} + \ldots + S_{2R-1}.$$

5152. (Professor Sylvester, F.R.S.)—If there be n homogeneous linear equations between $2n$ variables, and also n homogeneous linear relations between their reciprocals, prove that the resultant of the system of $2n$ equations is of the order 2^{2n-1}.

5154. (Professor Townsend, F.R.S.)—A cone of revolution of uniform density being supposed to float with vertex downwards and axis vertical in a gravitating fluid of greater density; show that the sphere having its axis for diameter will intersect its surface in the limiting circle of flotation consistent with the stability of its equilibrium under the action of gravity.

5157. (Professor Minchin, M.A.)—When a heavy body rests on a fixed rough surface, the equilibrium in the "neutral" case is said to be *unstable*. Show that the proof of this result is invalid, and that the true state of the case is as follows:—

The surfaces being both convex to each other, and ρ and ρ_1 being the radii of curvature of their sections (ρ_1 that of the body) made by the plane of displacement, the equilibrium will be stable or unstable according as

$$\frac{d\rho^{-1}}{ds} + \frac{d_1{}^{-1}\rho}{ds_1} \text{ is } - \text{ or } +;$$

but, if the contact takes place at a vertex of each, this condition fails, and then the equilibrium will be stable or unstable according as

$$\frac{d^2\rho^{-1}}{ds^2} + \frac{d^2\rho_1{}^{-1}}{ds_1{}^2} + \frac{(\rho+\rho_1)(\rho+2\rho_1)}{\rho^2\rho_1{}^2} \text{ is } - \text{ or } +,$$

ds and ds_1 being the elements of the arcs of the curves of section. The first condition may, of course, fail in other cases also.

5160. (Professor Evans, M.A.)—Find the number of real roots in the equation $a^x + bx^n = c$.

5161. (Editor.)—In a given triangle place (1) a given circle so that the area of the triangle polar to the given triangle, relatively to the given circle, may be a minimum; and (2) show that, when the area of the polar triangle is constant, the locus of the centre of the circle is a cubic hyperbola.

5166. (J. Richards, M.A.)—Place a chord of given length in a conic so that the lines joining its ends with two given points shall make with it equal angles.

5170. (R. Battle).—Through a fixed point in the axis of a conic draw a chord such that the rectangle or ratio of the lines joining its extremities with a given point shall be given.

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